MATH 3J04: Solutions to Home Assignment # 1

Problem 7.1 #14: The spectrum consists of a single eigenvalue $\lambda_1 = -2$ and a double eigenvalue $\lambda_2 = \lambda_3 = 0$. The eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \mathbf{x}_2 = \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Geometric multiplicity of the double eigenvalue is one, while algebraic multiplicity is two.

Problem 7.3 #12: The skew-symmetric matrices of 3x3, 5x5, etc. are always singular, while the skew-symmetric matrices of 2x2, 4x4, etc. may generally be non-singular. The simplest way to show this is to use the form,

$$\det(A) = (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n$$

The eigenvalues of skew-symmetric matrices are pure imaginary and complex conjugate. Therefore, if there is an odd number of eigenvalues ($n = 3, 5$,etc), one eigenvalue is always zero, i.e. $\det(A) = 0$.

Problem 7.5 #10: The basis of eigenvectors is

$$X = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

and the diagonal matrix of eigenvalues $D = X^{-1}AX$ is

$$D = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}$$

Problem 18.1 #12: The Gauss elimination algorithm shows that the system has no solutions.

Problem 18.2 #4: With no row interchanges, the LU factorization of A is

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix}.$$

The linear system has a unique solution: $x_1 = -1$, $x_2 = 2$, $x_3 = -1$.

Problem 18.3 #4: The iterative scheme is

$$x_1^{(m+1)} = \frac{1}{8}\left(-11.5 - 2x_2^{(m)} - x_3^{(m)}\right),$$

$$x_2^{(m+1)} = \frac{1}{6}\left(18.5 - x_1^{(m+1)} - 2x_3^{(m)}\right),$$

$$x_3^{(m+1)} = \frac{1}{5}\left(12.5 - 4x_1^{(m+1)}\right),$$

The scheme converges to an exact solution: $x_1 = -2.5$, $x_2 = 2$, $x_3 = 4.5$.