2.3 Conformal mapping

2.3.1 Theorems on conformal mapping

Definition: The transformation $z \mapsto w = f(z)$ is called conformal mapping if angles between two straight lines are preserved by the transformation.

Theorem: Let $z_0 \in D \subset \mathbb{C}$, where D is a domain of $f(z)$. The mapping $w = f(z)$ is conformal near $z = z_0$ if $f(z)$ is analytic at $z = z_0$ and $|f'(z_0)| \neq 0$.

Examples:

$$f(z) = az, \qquad a \in \mathbb{C}$$

$$f(z) = z^p, \qquad p > 1$$

Open Mapping Theorem: Let $z \in D$ be an open domain, where $w = f(z)$ is analytic. Then, $w \in R$ is an open range.

Riemann Mapping Theorem: Let $D \subset \mathbb{C}$ be an open domain. There exists an analytic function $f(z)$ that maps D into a unit circle $R = \{w \in \mathbb{C} : |w| < 1\}$, or equivalently, into an upper half-plane $R = \{w \in \mathbb{C} : \text{Im}(w) > 0\}$.

Inverse Function Theorem: Let $f(z)$ be analytic function near $z = z_0$ and $|f'(z_0)| \neq 0$. Then, $f(z)$ has a unique analytic inverse $f^{-1}(w)$ near $w = w_0 = f(z_0)$.

2.3.2 Recipe # 8: How to find conformal mapping $w = f(z)$ of an open simply-connected domain D

1. Map the boundary of D onto the w-plane.
2. Map a single point inside D onto the w-plane.
3. The range R is enclosed by the boundary on the w-plane and includes the single point.

Simplest conformal mappings:

1. Linear transformation

$$f(z) = az + b, \qquad a, b \in \mathbb{C}$$

2. Inversion

$$f(z) = \frac{1}{z}$$

3. Linear fractional transformation

$$f(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, \qquad ad - bc \neq 0$$

Examples:

- Cayley transform

$$f(z) = \frac{z - ia}{z + ia}, \qquad a > 0$$

- Mobius transform

$$f(z) = e^{i\theta} \frac{z - \alpha}{\bar{\alpha} z - 1}, \qquad |\alpha| < 1$$

- Zhukovski transform

$$f(z) = \frac{a}{2} \left(z + \frac{1}{z} \right), \qquad a \in \mathbb{R}$$

2.3.3 Construction of harmonic functions with conformal mappings

Theorem: Consider a harmonic function $\phi(x, y)$ that solves the Laplace equation:

$$\phi_{xx} + \phi_{yy} = 0, \qquad (x, y) \in D \subset \mathbb{R}^2.$$

Construct a conformal mapping $w = f(z)$ from $z \in D$ onto $w \in R$, such that $|f'(z)| \neq 0$ for $z \in D$ and

$$w = f(z) = u(x, y) + iv(x, y).$$

The function $\phi(x, y) = \Phi(u(x, y), v(x, y)) = \Phi(u, v)$ is harmonic and solves the Laplace equation:

$$\Phi_{uu} + \Phi_{vv} = 0, \qquad (u, v) \in R \subset \mathbb{R}^2$$

Poisson formula in the upper half-plane:

$$\phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\phi(s, 0)ds}{y^2 + (x - s)^2}, \quad x \in \mathbb{R}, \; y > 0$$

Poisson formula in the unit disk:

$$\phi(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)\phi(1, t)dt}{1 + r^2 - 2r\cos(t - \theta)}, \quad 0 \leq r < 1, \; 0 \leq \theta \leq 2\pi$$

2.3.4 Recipe # 9: How to find a harmonic function $\phi(x, y)$ in a domain D

1. Find a conformal mapping $w = f(z)$ that maps D into an upper half-plane.
2. Solve the Laplace equation in the upper half-plane for $\Phi(u, v)$.
3. Solution of the original problem is $\phi(x, y) = \Phi(u(x, y), v(x, y))$.