2 Applications of function of complex variable

2.1 Evaluation of Fourier and Laplace integrals

2.1.1 Recipe # 4: Evaluation of Fourier integrals

We consider the Fourier integral in the principal-value sense:

$$\int_{-\infty}^{\infty} f(x)e^{ikx}dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)e^{ikx}dx,$$

or equivalently:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\cos(kx)dx &= \mathrm{Re} \int_{-\infty}^{\infty} f(x)e^{ikx}dx \\ \int_{-\infty}^{\infty} f(x)\sin(kx)dx &= \mathrm{Im} \int_{-\infty}^{\infty} f(x)e^{ikx}dx \end{aligned}$$

We assume that the function $f(x)$ is not singular on $x \in \mathbb{R}$.

1. Check that $f(z) \to 0$ as $R \to \infty$ on the contour

$$\gamma_R = \{z \in \mathbb{C} : \quad z = Re^{i\theta}, \;\; 0 \le \theta \le \pi\}$$

2. Check that $f(z)$ has only isolated singularities in the upper half-plane and compute residue terms at each singularity

3. Compute the Fourier integral by the Residue Theorem:

$$\int_{-\infty}^{\infty} f(x)e^{ikx}dx = 2\pi i \sum_{\mathrm{Im}(z_j)>0} \mathrm{Res}\left[f(z)e^{ikz}; z_j\right]$$

Example:

$$\int_{-\infty}^{\infty} \frac{e^{ikx}dx}{x^2 + a^2}$$

2.1.2 Recipe # 5: Evaluation of integrals of rational functions

We consider the integral of a rational function:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx,$$

where $P(x), Q(x)$ are polynomials and $\deg(Q) \geq \deg(P) + 2$. We assume that the polynomial $Q(x)$ has no roots on $x \in \mathbb{R}$.

1. Find all roots of $Q(z)$ in the upper half-plane and compute residue terms at each root z_j
2. Compute the integral by the Residue Theorem:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{\mathrm{Im}(z_j) > 0} \mathrm{Res} \left[\frac{P(z)}{Q(z)}; z_j \right]$$

2.1.3 Modified recipe # 5: Evaluation of integrals of trigonometric functions

We consider the integral of a trigonometric function on the period:

$$\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$$

We assume that the function $f(\cos\theta, \sin\theta)$ is a rational function and has no singularities on $0 \leq \theta 2\pi$.

1. Parametrize the unit circle in a complex plane:

$$z = e^{i\theta}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad \sin\theta = \frac{z - z^{-1}}{2i}$$

2. Reduce the problem to an integral of a rational function in the unit disc and compute it by the Residue Theorem:

$$\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta = \int_{\gamma_0} \frac{dz}{iz} f\left(\frac{z^2 - 1}{2iz}, \frac{z^2 + 1}{2z} \right).$$

Examples:

$$\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}, \qquad \int_{-\infty}^{\infty} \frac{(1 + x^2) dx}{(1 + x^4)}, \qquad \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}, \quad |a| > |b|$$

2.1.4 Recipe # 6: Evaluation of inverse Laplace integrals

We consider the inverse Laplace integral:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{c-iR}^{c+iR} F(s)e^{st}ds,$$

where $t > 0$ and c is to the right of all singularities of $F(s)$.

1. Check that $F(s) \to 0$ as $R \to \infty$ on the contour

$$\gamma_R = \left\{ z \in \mathbb{C} : \quad z = c + Re^{i\theta}, \ \ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \right\}$$

2. Check that $F(s)$ has only isolated singularities to the left of $\mathrm{Re}(s) = c$ and compute residue terms at each singularity

3. Compute the inverse Laplace integral by the Residue Theorem:

$$f(t) = \sum_{\mathrm{Re}(s_j)<c} \mathrm{Res}\left[F(s)e^{st}; s_j\right]$$

Example:

$$F(s) = \frac{1}{s^2+a^2}, \quad F(s) = \frac{1}{s^n}$$

Remark:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds = 0$$

for $t < 0$.