1.3 Laurent series and the Residue Theorem

1.3.1 Taylor series for analytic functions

Theorem: Assume that the function $f(z)$ is analytic near a point $z = z_0$ in the disk $|z - z_0| < R$. Then, the function $f(z)$ can be represented by the Taylor series for any $z \in \mathbb{C}: \quad |z - z_0| < R$:

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + ... \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\, f^{(k)}(z_0)\, (z - z_0)^k. \end{aligned}$$

Examples:

$$f(z) = \frac{1}{z^2 + 1}, \qquad f(z) = e^{-z^2}$$

Remarks:

1. The radius R of convergence of the Taylor series can be estimated from the D'Alembert ratio test:

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

where a_n is the coefficient of the Taylor series.

2. Taylor series for entire functions have $R = \infty$

$$f(z) = P_n(z), \qquad f(z) = e^z, \cos z, \sin z$$

3. Taylor series for singular functions at $z = z_0$ have $R = 0$

$$f(z) = \frac{\cos z}{z}, \qquad f(z) = e^{1/z}$$

1.3.2 Laurent series

Theorem: Assume that the function $f(z)$ is analytic in the annulus $R_1 < |z-z_0| < R_2$. Then, the function $f(z)$ can be represented by the Laurent series:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k,$$

where

$$c_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{k+1}},$$

and γ is any contour in the annulus.

Examples:

$$f(z) = \frac{1}{z^2+1}, \qquad f(z) = \frac{1}{(z-1)(z-2)}$$

Remarks:

1. The Laurent series converges absolutely in the annulus $R_1 < |z - z_0| < R_2$, where R_1 and R_2 can be estimated from the D'Alembert ratio test:
$$R_2 = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|, \qquad R_1 = \lim_{n\to\infty} \left| \frac{c_{-n-1}}{c_{-n}} \right|.$$
2. When $R_1 > 0$, the function $f(z)$ may have non-isolated singularities at $z = z_0$ Taylor series for entire functions have $R = \infty$
$$f(z) = \frac{1}{\sqrt{1-z^2}}$$
3. When all $c_n = 0$ for $n \leq -1$, the function $f(z)$ is regular at $z = z_0$. When all $c_n = 0$ for $n \geq 1$, the function $f(z)$ is regular at infinity $z = \infty$.
4. When $R_1 = 0$, the point $z = z_0$ is either regular or isolated singularity for $f(z)$. When $R_2 = \infty$, the point $z = \infty$ is either regular or isolated singularity.

1.3.3 Properties of isolated singularities

1. Pole singularity

   - The point $z = z_0$ is a *pole of order* N for the function $f(z)$ if
   $$f(z) = \frac{\phi(z)}{(z - z_0)^N}$$
   where $\phi(z)$ is analytic at $z = z_0$.
   - If the function $f(z)$ has a pole of order N at $z = z_0$, the Laurent series at $z = z_0$ has all $c_n = 0$ for $n \leq -N - 1$.
   - If the function $f(z)$ has only pole singularities in $z \in \mathbb{C}$, it is called a *meromorphic* function

2. Essential singularity

   - If the point $z = z_0$ is an isolated (non-removable) singularity of $f(z)$ and it is not a pole, it is an *essential singularity.*
   - If the function $f(z)$ has an essential singularity at $z = z_0$, the Laurent series at $z = z_0$ has some or all $c_n \neq 0$ for $n \leq -N - 1$.
   - If the function $f(z)$ has an essential singularity at $z = z_0$, then $f(z)$ pass arbitrary close to any complex number in the neighborhood of $z = z_0$.

Examples:

$$f(z) = \frac{z}{\sin z}, \qquad f(z) = e^{1/z}$$

1.3.4 The Residue Theorem

The coefficient c_{-1} in the Laurent series of $f(z)$ at $z = z_0$ is called the *residue* of $f(z)$ at $z = z_0$:

$$\text{Res}[f(z); z_0] = \frac{1}{2\pi i} \int_\gamma f(\zeta) d\zeta$$

Theorem: Let $f(z)$ be analytic inside a closed contour γ, except for isolated singularities at $z = \{z_1, z_2, ..., z_n\}$. Then,

$$\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f(z); z_k]$$

Examples:

$$\int_\gamma z^n e^{1/z} dz, \qquad \int_\gamma \frac{dz}{1 + 4z^2}$$

1.3.5 Recipe #3: Evaluation of contour integrals with calculus of residues

$$\int_\gamma f(z)dz$$

where $f(z)$ has isolated singularities inside γ.

1. Find all isolated singularities of $f(z)$ inside γ. Check that no non-isolated singularities of $f(z)$ inside γ exist.

2. For each isolated singularity, find the residue term of $f(z)$.

   (a) Let $z = z_0$ be a simple zero of $Q(z)$ and $f(z) = \frac{P(z)}{Q(z)}$. Then

   $$\text{Res}[f(z); z_0] = \frac{P(z_0)}{Q'(z_0)}$$

   (b) Let $z = z_0$ be a simple pole. Then

   $$\text{Res}[f(z); z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$

   (c) Let $z = z_0$ be a pole of order N. Then

   $$\text{Res}[f(z); z_0] = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}} \left[(z - z_0)^N f(z) \right]$$

   (d) Let $z = z_0$ be a point of essential singularity. Then, compute the Laurent series of $f(z)$ at $z = z_0$ and find c_{-1}.

3. Sum all residue terms.

Examples:

$$f(z) = \cot z, \qquad f(z) = \frac{z^2 + 2z}{(z-1)^3}, \qquad f(z) = \sin\left(\frac{1}{z}\right).$$