1.2 Cauchy integral formulas

1.2.1 Contour integration in a complex plane

Consider a function $f(z)$ that depends on $z = x + iy$ and does not depend on $\bar{z} = x - iy$. Consider a contour γ in $z \in \mathbb{C}$ that is parameterized by a real parameter t, such that

$$\gamma : \quad z = z(t), \quad x = x(t), y = y(t), \quad t_a \leq t \leq t_b$$

The contour integral of $f(z)$ over the contour γ is

$$\int_\gamma f(z)dz = \int_{t_a}^{t_b} f(z(t))z'(t)dt = \int_\gamma (udx - vdy) + i \int_\gamma (udy + vdx)$$

Example:

$$f(z) = z, \qquad f(z) = \bar{z} \qquad \text{on a unit circle}$$

Properties of the contour γ in our studies:

1. a simply-connected curve on $\mathbb{C}$
2. either open or closed curve on $\mathbb{C}$
3. $z(t)$ is continuously differentiable on $t_a \leq t \leq t_b$
4. parameter t increases in the counter-clockwise direction which is referred to as a *positive* direction

Theorem: If $f(z) = F'(z)$ in $z \in D \subset \mathbb{C}$. Then, for any $\gamma \in D$, it is true that

$$\int_\gamma f(z)dz = F(z(t_b)) - F(z(t_a)).$$

1.2.2 The Cauchy Integral Theorem

Theorem: Assume that the function $f(z)$ is analytic in $z \in D \subset \mathbb{C}$. Then,

$$\int_{\gamma} f(z)dz = 0$$

for any closed contour $\gamma \in D$.

Remarks:

1. Let γ be an open contour in D. The integral $\int_{\gamma} f(z)dz$ between two points $z = a$ and $z = b$ in $z \in D$ does not depend on the path of integration between $z = a$ and $z = b$.
2. Let γ be a closed contour in D. The integral $\int_{\gamma} f(z)dz$ is zero if $f(z)$ does not have singularities inside and on the contour $\gamma \subset D$. It can be non-zero if $f(z)$ has a singularity inside γ.

Examples:

$$f(z) = z^n, \qquad f(z) = \frac{1}{z^n} \qquad \text{on the unit circle}$$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{P'(z)dz}{P(z)} = k,$$

where $P(z)$ is a polynomial of any degree and k is the number of roots of $P(z)$ inside γ.

Inverse Theorem: If $\int_{\gamma} f(z)dz = 0$ for any closed contour $\gamma \in D \subset \mathbb{C}$ and $f(z)$ is continuous function in $z \in D$, then $f(z)$ is analytic in $z \in D$.

1.2.3 The Cauchy Integral Formula

Theorem: Assume that the function $f(z)$ is analytic in $z \in D \subset \mathbb{C}$. Then,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} = f(z), \qquad \text{for any } z \text{ inside } \gamma$$

for any closed contour $\gamma \in D$.

Theorem: Assume that the function $f(z)$ is analytic in $z \in D \subset \mathbb{C}$. Then,

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{k+1}}, \qquad \text{for any } z \text{ inside } \gamma, \quad k \geq 1,$$

for any closed contour $\gamma \in D$.

Remarks:

1. If the function $f(z)$ is analytic in $z \in D \subset \mathbb{C}$, then any higher-order derivative of $f(z)$ in $z \in D$ is continuous, i.e. $f(z) \in C^\infty(D)$.
2. If the function $f(z)$ has isolated singularities at the points z_1,z_2,...,z_n in D and is analytic in $z \in D \backslash \{z_1, z_2, ..., z_n\}$, then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + ... + \int_{\gamma_n} f(z)dz$$

   where contour γ surrounds $\{z_1, z_2, ..., z_n\}$ and the contour γ_j surrounds the point z_j.

1.2.4 Recipe #2: Evaluation of contour integrals with Cauchy formulas

$$\int_\gamma \frac{f(z)dz}{g(z)}$$

where $f(z)$ and $g(z)$ are analytic inside γ.

1. Find all zeros of $g(z)$ inside γ
2. If no zeros of $g(z)$ inside γ exist, the integral equals to zero by the Cauchy Integral Theorem
3. Assume that there exists a single zero inside γ. If the zero is simple, evaluate the integral by the Cauchy Integral Formula with $k = 1$. If the zero is multiple, use the second Cauchy Integral Formula with $k \geq 2$.
4. Assume that there exist several zeros inside γ. Split the integral over the contour γ into the sum of the integrals over several contours, each surrounds one zero. Repeat step 3 for each individual sub-integral.

Examples:

$$\int_\gamma \frac{e^z \cos z^2 dz}{z(z^2-5)}, \qquad \int_\gamma \frac{e^z dz}{(z-i)^3}$$

$$\frac{1}{2\pi i}\int_\gamma \frac{Q(z)dz}{P(z)} = \sum_{k=1}^{n} \frac{Q(z_k)}{P'(z_k)},$$

where $Q(z)$ and $P(z)$ are polynomials of any degree and $\{z_1, z_2, ..., z_n\}$ are *simple* zeros of $P(z)$ inside γ.