5.2 Laplace transform and applications

5.2.1 Theorem on the Laplace transform

Theorem: Let $f(t)$ be a continuous function on $t \geq 0$, which may grow at most exponentially at infinity:

$$\exists \alpha > 0, \ \ |f_\infty| < \infty : \qquad \lim_{t \to +\infty} e^{-\alpha t} f(t) = f_\infty.$$

Then, the Laplace transform $F(s)$ exists for any $s \in \mathbb{R}$:

$$F(s) = \int_0^\infty f(t) e^{-st} dt.$$

The function $f(t)$ can be replaced by the inverse Laplace transform:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} ds, \qquad c > \alpha$$

where $c \in \mathbb{R}$ is located to the right of any singularities of $F(s)$.

Examples:
Exponential function:

$$f(t) = e^{\alpha t}, \qquad F(s) = \frac{1}{s - \alpha}$$

Power function:

$$f(t) = t^n, \qquad F(s) = \frac{n!}{s^{n+1}}$$

Trigonometric functions:

$$f(t) = \sin(\omega t), \qquad F(s) = \frac{\omega}{\omega^2 + s^2}$$

$$f(t) = \cos(\omega t), \qquad F(s) = \frac{s}{\omega^2 + s^2}$$

5.2.2 Properties of the Laplace transform

- Laplace transform is a linear operator

$$\begin{aligned} (i) \quad & \mathcal{L}[f(t)+g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] \\ (ii) \quad & \mathcal{L}[\lambda f(t)] = \lambda \mathcal{L}[f(t)] \end{aligned}$$

- exponential factor

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

- shift

$$g(t) = \begin{cases} 0, & 0 \leq t \leq a \\ f(t-a), & t \geq a \end{cases} \qquad \mathcal{L}[g(t)] = e^{-as} F(s)$$

- first derivative

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

- second derivative

$$\mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

- convolution

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}\left[\int_0^t f(\tau) g(t-\tau) d\tau\right] = F(s)G(s)$$

5.2.3 Recipe # 15: Solutions of PDEs with the Laplace transform

$$u_t = u_{xx}, \qquad x \geq 0, \;\; t \geq 0$$

such that

$$u(0,t) = f(t), \qquad t \geq 0$$

and

$$u(x,0) = 0, \qquad x \geq 0$$

where $f(t)$ is continuous and bounded for any $t \geq 0$.

1. Apply Laplace transform to $f(t)$:

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

2. Apply the time-dependent Laplace transform to $u(x,t)$:

$$U(x,s) = \int_0^{\infty} u(x,t)e^{-st}dt$$

such that $U(0,s) = F(s)$.

3. Apply Laplace transform to the PDE and obtain the initial-value problem for $U(x,s)$ in x:

$$\mathcal{L}[u_t(x,t)] = sU(x,s) - U(x,0), \qquad \mathcal{L}[u_{xx}(x,t)] = \frac{\partial^2 U(x,s)}{\partial x^2},$$

4. Find a unique solution for $U(s,t)$:

$$U(s,t) = F(s)e^{-x\sqrt{s}}$$

5. Invert the Laplace transform and obtain a unique solution for $u(x,t)$:

$$u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{-x\sqrt{s}}e^{st}ds$$