4.4 Characteristics of second-order PDEs

Consider the second-order PDE with constant coefficients:

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0$$

where a,b,c are constants and $u = u(x, y)$.

Try a particular solution $u(x, y)$ in the form:

$$u(x, y) = U(x - \lambda y) : \quad a - 2b\lambda + c\lambda^2 = 0$$

4.4.1 Classification of second-order PDEs

- Hyperbolic PDE, if λ_1, λ_2 are real and distinct

$$\text{(wave)} \quad u_{tt} - c^2 u_{xx} = 0$$

- Elliptic PDE, if λ_1, λ_2 are complex-conjugate

$$\text{(Laplace)} \quad u_{xx} + u_{yy} = 0$$

- Parabolic PDE, if $\lambda_1 = \lambda_2$ are real multiple

$$\text{(heat)} \quad u_{xx} = u_t$$

4.4.2 Explicit solutions by characteristics

- D'Alembert solution of the wave equation

$$u(x, t) = F(x - ct) + G(x + ct)$$

  where $F(x)$ and $G(x)$ are found from initial conditions at $t = 0$

- Complex function solution of the Laplace equation

$$u(x, y) = F(x + iy) + G(x - iy) = F(x + iy) + \overline{F(x + iy)}$$

  where $F(z)$ is any analytic function of $z = x + iy$.