4.2 Laplace equation in two dimensions

4.2.1 Typical domains in $(x, y) \in \mathbb{R}^2$

- rectangle in rectangular coordinates (x, y)

  $$\Delta u = u_{xx} + u_{yy}$$

  in

  $$D = \{(x, y) : \ \ 0 \le x \le a, \ \ 0 \le y \le b\}$$

- disc in polar coordinates (r, θ)

  $$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

  in

  $$D = \{(r, \theta) : \ \ 0 \le r \le a, \ \ 0 \le \theta \le 2\pi\}$$

  where $x = r\cos\theta$ and $y = r\sin\theta$

4.2.2 Modified recipe #12: for time-dependent problems

$$u_{tt} = c^2 \Delta u, \qquad (x, y) \in D, \ \ t \ge 0$$
$$u_t = k\Delta u, \qquad (x, y) \in D, \ \ t \ge 0$$

1. Use the product form by separating time t from space (x, y):

   $$u(x, y, t) = \phi(x, y)T(t)$$

   where $\phi(x, y)$ solves the Helmholtz equation:

   $$\Delta\phi + \lambda\phi = 0$$

2. Use the product form for separating one space variable from another space variables:

   $$\phi(x, y) = X(x)Y(y), \qquad \phi(r, \theta) = R(r)\Theta(\theta)$$

3. Solve two Sturm–Liouville problems with two eigenvalue parameters λ and μ and two sets of eigenfunctions.

4. Apply steps 3-5 of recipe # 11 by replacing series of eigenfunctions with double series of eigenfunctions.

4.2.3 Important theorems for cylindrical harmonics

Theorem: Let $f(\theta)$ be a periodic continuously differentiable function with period 2π. The function $f(\theta)$ can be replaced by the complex Fourier series:

$$f(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta},$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta, \qquad n \in \mathbb{Z},$$

such that the series converges uniformly to $f(\theta)$ on $0 \leq \theta \leq 2\pi$.

Theorem: Let $g(r)$ be continuously differentiable function on $0 \leq r \leq a$, such that $g(0)$ exists and $g(a) = 0$. The function $g(r)$ can be replaced by the Bessel series:

$$g(r) = \sum_{m=1}^{\infty} d_m J_n\left(\frac{z_{n,m}r}{a}\right), \qquad \forall n \in \mathbb{Z},$$

where

$$d_m = \frac{2}{a^2 J_{n+1}^2(z_{n,m})}\int_0^a r g(r) J_n\left(\frac{z_{n,m}r}{a}\right) dr, \qquad m \geq 1,$$

such that the series converges uniformly to $g(r)$ on $0 \leq r \leq a$, where $z_{n,m}$ are roots of $J_n(z) = 0$.

Eigenfunctions $u_{n,m}(r,\theta) = J_n\left(\frac{z_{n,m}r}{a}\right)e^{in\theta}$, $n \in \mathbb{Z}$, $m \geq 1$ are referred to as the cylindric harmonics in a disc $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$.