3.3 Fourier series expansions

3.3.1 List of important theorems

Theorem: Let $f(x)$ be a piecewise continuously differentiable function on the interval $-l \leq x \leq l$. Then, it can be represented by the Fourier series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n x}{l} + b_n \sin \frac{\pi n x}{l} \right),$$

where

$$\begin{aligned} a_n &= \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{\pi n x}{l} dx, \quad n = 0, 1, 2, ... \\ b_n &= \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{\pi n x}{l} dx, \quad n = 1, 2, ... \end{aligned}$$

The Fourier series converges to $f(x)$ for any x, where $f(x)$ is continuous, and to $\frac{1}{2}(f(x+0) + f(x-0))$ for any x, where $f(x)$ has a jump discontinuity.

Theorem: Let $f(x)$ be an even (piecewise continuously differentiable) function of x. Then, it can be represented on the interval $0 \leq x \leq l$ by the Fourier cosine series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{\pi n x}{l},$$

where

$$a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{\pi n x}{l} dx, \quad n = 0, 1, 2, ...$$

Theorem: Let $f(x)$ be an odd (piecewise continuously differentiable) function of x. Then, it can be represented on the interval $0 \le x \le l$ by the Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{\pi n x}{l},$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{\pi n x}{l} dx, \quad n = 1, 2, \ldots$$

Theorem: Let $f(x)$ be a continuously differentiable function on the interval $-l \le x \le l$, such that

$$f(-l) = f(l), \qquad f'(-l) = f'(l).$$

The Fourier series converges uniformly to $f(x)$ on $-l \le x \le l$, such that

$$\lim_{N \to \infty} \max_{-l \le x \le l} |f(x) - f_N(x)| = 0,$$

where $f_N(x)$ is N-th partial sum (trigonometric approximation)

$$f_N(x) = \frac{1}{2} a_0 + \sum_{n=1}^{N} \left(a_n \cos \frac{\pi n x}{l} + b_n \sin \frac{\pi n x}{l} \right).$$

Remarks:

- If Fourier series converges uniformly, it can be differentiated term-by-term.
- If the Fourier series converges in a pointwise sense, it can be integrated term by term.
- If the function $f(x)$ has a jump, the Fourier series do not converge uniformly and the N-th partial sum oscillates near the jump points (Gibbs oscillations)

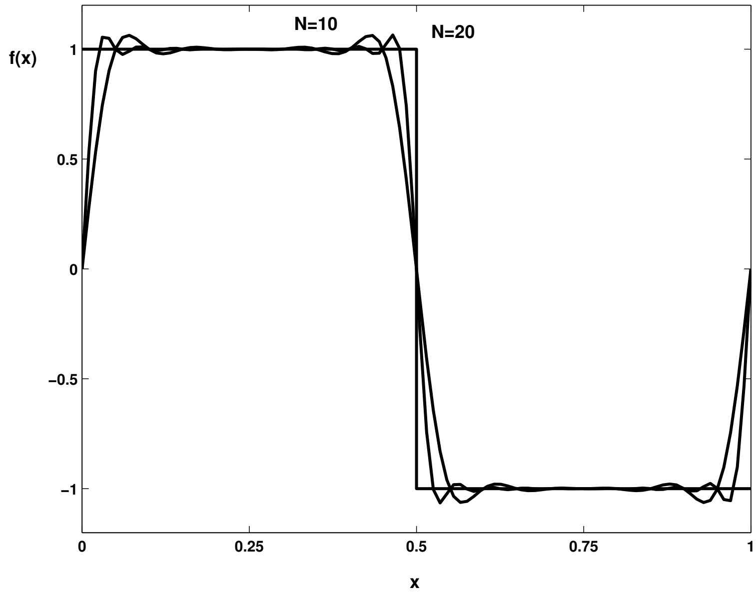

Figure 1: Gibbs phenomenon for non-uniform Fourier series.

3.3.3 List of particular Fourier series

$$|x| = \frac{1}{2} - \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos\pi(2m-1)x}{(2m-1)^2}, \qquad -1 \leq x \leq 1$$

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\pi n x}{n}, \qquad -1 \leq x \leq 1$$

$$\mathrm{sign}(x) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin\pi(2m-1)x}{2m-1}, \qquad -1 \leq x \leq 1$$

3.3.3 Three faces of Fourier series

- Full Fourier series

  1. Series of eigenfunctions of $\mathcal{A} = -\frac{d^2}{dx^2}$ on $-l \leq x \leq l$ with periodic boundary conditions in $L^2([-l, l])$.
  2. Converges pointwisely to $f(x)$ at any value of x, where $f(x)$ is continuous, if $f(x)$ is piecewise continuous.
  3. Converges uniformly if $f(x)$ is continuous on $-l \leq x \leq l$ and satisfies the periodic boundary conditions: $f(-l) = f(l)$ and $f'(-l) = f'(l)$.

- Fourier sine-series

  1. Series of eigenfunctions of $\mathcal{A} = -\frac{d^2}{dx^2}$ on $0 \leq x \leq l$ with Dirichlet boundary conditions in $L^2([0, l])$.
  2. Converges pointwisely to $f(x)$ at any value of x, where $f(x)$ is continuous, if $f(x)$ is piecewise continuous.
  3. Converges uniformly if $f(x)$ is continuous on $0 \leq x \leq l$ and satisfies the Dirichlet boundary conditions: $f(0) = f(l) = 0$.

- Fourier cosine-series

  1. Series of eigenfunctions of $\mathcal{A} = -\frac{d^2}{dx^2}$ on $0 \leq x \leq l$ with Neumann boundary conditions in $L^2([0, l])$.
  2. Converges pointwisely to $f(x)$ at any value of x, where $f(x)$ is continuous, if $f(x)$ is piecewise continuous.
  3. Converges uniformly if $f(x)$ is continuous on $0 \leq x \leq l$ and satisfies the Neumann boundary conditions: $f'(0) = f'(l) = 0$.