2.3. Power series solutions of second-order ODEs

Consider the linear homogeneous second-order equation:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are given smooth functions of $x \in \mathbb{R}$

2.3.1. Classification of points in the second-order ODE

The point $x = 0$ is:

- *ordinary* if $p(0)$ and $q(0)$ exist
- *regular singular* if $p(0)$ or $q(0)$ do not exist but $\lim_{x\to 0} xp(x)$ and $\lim_{x\to 0} x^2q(x)$ exist
- *irregular singular* if $\lim_{x\to 0} xp(x)$ or $\lim_{x\to 0} x^2q(x)$ do not exist

Examples:

$$\begin{aligned} y'' + y &= 0 \\ x^2y'' + y &= 0 \\ x^4y'' + y &= 0 \end{aligned}$$

2.3.2. List of important theorems

Theorem: Let $x = 0$ be an ordinary point. Two linearly independent solutions $y_1(x)$ and $y_2(x)$ can be found in the power series form:

$$y_{1,2}(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where $\{a_k\}_{k=0}^{\infty}$ are numerical coefficients. The power series converge for $|x| < R$, where $R > 0$ is a distance to the closest singular point.

Theorem: Let $x = 0$ be a regular singular point. At least one solution $y_1(x)$ can be found in the Frobenius series form:

$$y_1(x) = x^{p_1} \sum_{k=0}^{\infty} a_k x^k,$$

where $\{a_k\}_{k=0}^{\infty}$ are numerical coefficients and p_1 is the largest index of the Euler singular equation. The Frobenius series converges for $|x| < R$, where $R > 0$ is a distance to the closest singular point.

Remarks:

- If $p_1 - p_2$ is not an integer, the other solution $y_2(x)$ can also be found in the Frobenius series form.
- If $p_1 - p_2$ is an integer, the second solution has a modified Frobenius series form:

  $$y_2(x) = c_0 \ln|x| y_1(x) + x^{p_2} \sum_{k=0}^{\infty} b_k x^k,$$

  where $\{b_k\}_{k=0}^{\infty}$ are numerical coefficients and c_0 is constant, which can be zero.
- If $p_1 = p_2$, the second solution $y_2(x)$ has always a logarithmic term (i.e. $c_0 \neq 0$)

2.3.3. Recipe # 9: Power series solution of second-order ODEs

Assume that $x = 0$ is an ordinary point of the second-order ODE:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

1. Look for power series solution near $x = 0$:

$$y(x) = \sum_{k=0}^{\infty} a_k x^k,$$

2. Find a recursive equation for numerical coefficients $\{a_k\}_{k=0}^{\infty}$
3. Identify two linearly independent solutions for $y_1(x)$ and $y_2(x)$
4. Find radius R of convergence of the power series solution
5. Find an explicit expression for a_n, $n = 0, 1, ...$ for each solution, if possible
6. Find closed analytical expressions for the fundamental solutions $y_1(x)$ and $y_2(x)$, if possible

Example: the harmonic oscillation equation

$$y'' + \omega_0^2 y = 0$$

2.3.4. Recipe # 10: Frobenius series solution of second-order ODEs

Assume that $x = 0$ is a regular singular point of the second-order ODE:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

1. Find the indices p_1 and p_2 ($p_1 \geq p_2$) of the Euler singular equation
2. Look for Frobenius series solution near $x = 0$:
$$y(x) = x^{p_1} \sum_{k=0}^{\infty} a_k x^k,$$
3. Find a recursive equation for numerical coefficients $\{a_k\}_{k=0}^{\infty}$
4. Find radius R of convergence of the Frobenius series solution
5. Find an explicit expression for a_n, $n = 0, 1, ...,$ if possible
6. Find a closed analytical expression for $y_1(x)$, if possible
7. Construct the second solution $y_2(x)$ from a Wronskian equation and a modified Frobenius series

Example: the Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0, \qquad n \in \mathbb{N}, \;\; x \geq 0$$