2. Ordinary differential equations

2.1. Linear equations with constant coefficients

2.1.1. Recipe # 4: Solution of homogeneous systems of differential equations

Consider the homogeneous system:

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y},$$

where $\mathbf{y} \in \mathbb{R}^n$, A is constant n-by-n matrix, and $\mathbf{y}(0) = \mathbf{y}_0$.

1. Look for particular solutions by separating the variables:

$$\mathbf{y}(t) = e^{\lambda t}\mathbf{x} : \;\; A\mathbf{x} = \lambda\mathbf{x}$$

2. Find all eigenvalues and eigenvectors of A:

$$A\mathbf{u}_j = \lambda_j \mathbf{u}_j, \quad j = 1, 2, ..., n$$

3. If eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ are linearly independent (form a basis in $\mathbb{R}^n$), then construct a general solution by the Linear Superposition Principle:

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2 + ... + c_n e^{\lambda_n t}\mathbf{u}_n$$

4. Find the unique solution from the initial value:

$$\mathbf{y}(0) = \mathbf{u}_1 c_1 + \mathbf{u}_2 c_2 + ... + \mathbf{u}_n c_n = \mathbf{y}_0,$$

which is an inhomogeneous linear system.

Example:

$$\begin{aligned} \frac{dy_1}{dt} &= 2y_1 + 2y_2 \\ \frac{dy_2}{dt} &= 5y_1 - y_2 \end{aligned}$$

2.1.2. Recipe # 5: Diagonalization of homogeneous systems of ODEs

Consider the homogeneous system:

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y},$$

where $\mathbf{y} \in \mathbb{R}^n$, A is constant n-by-n matrix, and $\mathbf{y}(0) = \mathbf{y}_0$.

1. Find the basis eigenvectors of A:

   $$A\mathbf{u}_j = \lambda_j \mathbf{u}_j, \quad j = 1, 2, ..., n$$

   such that $S = [\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n]$ is non-singular, e.g. $\det(S) \neq 0$

2. Apply the similarity transformation:

   $$\mathbf{y}(t) = S\mathbf{z}(t) : \qquad \frac{d\mathbf{z}}{dt} = D\mathbf{z},$$

   where D is a diagonal matrix of eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$

3. Solve the uncoupled first-order ODEs:

   $$\frac{dz_j}{dt} = \lambda_j z,$$

   such that $z_j(t) = c_j e^{\lambda_j t}$

4. Construct a general solution by using the similarity transformation:

   $$\mathbf{y}(t) = S\mathbf{z}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2 + ... + c_n e^{\lambda_n t}\mathbf{u}_n$$

5. Find the unique solution from the initial value:

   $$\mathbf{y}(0) = \mathbf{u}_1 c_1 + \mathbf{u}_2 c_2 + ... + \mathbf{u}_n c_n = S\mathbf{c} = \mathbf{y}_0,$$

   such that $\mathbf{c} = S^{-1}\mathbf{y}_0$.

Example:

$$\begin{aligned} \frac{dy_1}{dt} &= 2y_1 + 2y_2 \\ \frac{dy_2}{dt} &= 5y_1 - y_2 \end{aligned}$$

Coordinates $\{z_1(t), z_2(t), ..., z_n(t)\}$ in basis $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ are called normal coordinates.

2.1.3. Recipe # 6: Solution of systems of second-order equations

Consider the second-order homogeneous system:

$$M\frac{d^2\mathbf{y}}{dt^2} + A\mathbf{y} = \mathbf{0},$$

with the initial values:

$$\mathbf{y}(0) = \mathbf{y}_0, \qquad \mathbf{y}'(0) = \mathbf{p}_0,$$

where $\mathbf{y} \in \mathbb{R}^n$, A is constant n-by-n matrix, M is a constant n-by-n diagonal matrix with positive coefficients.

1. Look for particular solutions by separating the variables:

$$\mathbf{y}(t) = \cos(\omega t + \delta)\mathbf{x} : \; A\mathbf{x} = \omega^2 M\mathbf{x}$$

2. Find all eigenvalues and eigenvectors of the generalized eigenvalue problem:

$$A\mathbf{u}_j = \lambda_j M\mathbf{u}_j, \quad j = 1, 2, ..., n$$

   where $\lambda_j = \omega_j^2$

3. If eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ form a basis in $\mathbb{R}^n$, then construct a general solution by the Linear Superposition Principle:

$$\mathbf{y}(t) = c_1\cos(\omega_1 t+\delta_1)\mathbf{u}_1 + c_2\cos(\omega_2 t+\delta_2)\mathbf{u}_2 + ... + c_n\cos(\omega_n t+\delta_n)\mathbf{u}_n$$

4. Find the unique solution from the initial values:

$$\mathbf{y}(0) = \mathbf{u}_1 c_1\cos\delta_1 + \mathbf{u}_2 c_2\cos\delta_2 + ... + \mathbf{u}_n c_n\cos\delta_n = \mathbf{y}_0,$$

$$\mathbf{y}'(0) = -\mathbf{u}_1\omega_1 c_1\sin\delta_1 - \mathbf{u}_2\omega_2 c_2\sin\delta_2 - ... - \mathbf{u}_n\omega_n c_n\sin\delta_n = \mathbf{p}_0.$$

Example:

$$\begin{aligned} m\frac{d^2y_1}{dt^2} + ky_1 + k(y_1 - y_2) &= 0 \\ m\frac{d^2y_2}{dt^2} + k(y_2 - y_1) + ky_2 &= 0 \end{aligned}$$

Eigenvalues $\{\omega_1, \omega_2, ..., \omega_n\}$ are normal frequencies of oscillations
Eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ are normal modes of oscillations