Theorem

Let a be any number. Any polynomial $p(x) \in P_n$ can be uniquely represented by

$$p(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n.$$

Example

$$a = 2, \qquad p(x) = x^3 - 2x^2 + x - 1$$

Theorem

Let $p(x)$ be a polynomial of degree $n \geq 1$ and a be any number. Then $p(x) = p(a) + (x-a)q(x)$, where $q(x)$ is a polynomial of degree $n-1$. Moreover, $p(a) = 0$ if and only if $p(x) = (x-a)q(x)$.

Example

$$p(x) = x^3 - x^2 + x - 1$$

Taylor Theorem

Let $p(x)$ be a polynomial of degree n. Then,

$$p(x) = p(a) + p'(a)(x-a) + \frac{1}{2!}p''(a)(x-a)^2 + \cdots + \frac{1}{n!}p^{(n)}(a)(x-a)^n.$$

Example

Consider two subspaces of polynomials

$$p(x) = (1+x)^n$$

Example

Find the polynomial $p(x)$ in P_3 such that

$$p(0) = 0, \quad p(1) = 1, \quad p(2) = 2, \quad p(3) = 9$$

Lagrange polynomials

Let $\{x_0, x_1, ..., x_n\}$ be the set of distinct x values. Let $\{\delta_0(x), \delta_1(x), \cdots, \delta_n(x)\}$ be a set of polynomials in P_n, such that

$$\delta_k(x_k) = 1, \quad \delta_k(x_j) = 0, \quad j \neq k,$$

and $k = 0, 1, ...n$. These polynomials are called Lagrange polynomials and they can be found explicitly as

$$\delta_k(x) = \frac{(x - x_0)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)}{(x_k - x_0)...(x_k - x_{k-1})(x_k - x_{k+1})...(x_k - x_n)}.$$

Theorem The set of Lagrange polynomials is a basis in P_n, such that any polynomial $p(x) \in P_n$ can be expressed uniquely as

$$p(x) = p(x_0)\delta_0(x) + p(x_1)\delta_1(x) + ... + p(x_n)\delta_n(x).$$

Fundamental Algebra Theorem

Every polynomial of degree $n \geq 1$ has exactly n roots (perhaps, multiple or complex-valued), such that

$$p(x) = c(x - x_1)(x - x_2) \cdots (x - x_n)$$

Example

$$p(x) = x^3 - x^2 + x - 1$$

Theorem

Let $p(x)$ be a polynomial in P_n with $(n + 1)$ distinct roots. Then, $p(x) \equiv 0$.