Any matrix multiplication is a linear transformation. Linear transformations are uniquely defined by matrices.

Theorem

Let $T : \mathbf{R}^n \mapsto \mathbf{R}^m$ be a linear transformation. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a standard basis in $\mathbf{R}^n$. Then T is uniquely defined by the matrix $A = [T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)]$.

Example

Rotation in $\mathbf{R}^2$ onto a counterclockwise direction around the origin by the angle θ:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Matrices of linear transformations can be introduced for any bases, not only for standard basis.

Example

Let $T : \mathbf{R}^2 \mapsto \mathbf{R}^2$ be a linear transformation and

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-2\\4\end{pmatrix}, \qquad T\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$$

Compute

$$T\begin{pmatrix}4\\3\end{pmatrix}$$

Let $T : \mathbf{R}^n \mapsto \mathbf{R}^m$ and $S : \mathbf{R}^m \mapsto \mathbf{R}^k$ be two linear transformations. The **composite** transformation $S \circ T$ is equivalent to a repeated matrix multiplication of the corresponding matrices for T and S:

$$S \circ T = \mathbf{R}^n \mapsto \mathbf{R}^k,$$

such that

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u})) \quad \text{for all} \quad \mathbf{u} \in \mathbf{R}^n.$$

Example

Let ϕ and θ be two angles and let R_ϕ and R_θ be two rotations in $\mathbf{R}^2$. Find the transformation $R_\theta \circ R_\phi = R_{\theta+\phi}$.

Let $T : \mathbf{R}^n \mapsto \mathbf{R}^m$ and $S : \mathbf{R}^m \mapsto \mathbf{R}^n$ be two transformations, such that

$$(S \circ T)(\mathbf{u}) = \mathbf{u} \quad \text{for all} \quad \mathbf{u} \in \mathbf{R}^n$$

and

$$(T \circ S)(\mathbf{u}) = \mathbf{u} \quad \text{for all} \quad \mathbf{u} \in \mathbf{R}^m.$$

Then T is **invertible** and S is the **inverse** of T: $S = T^{-1}$.

<u>Theorem</u>

T is invertible if and only if the corresponding matrix A is invertible. The transformation matrix for the inverse of T is A^{-1}.

<u>Example</u>

Find the inverse of the counterclockwise rotation $R_\theta : \mathbf{R}^2 \mapsto \mathbf{R}^2$.

In order to find the inverse matrix A^{-1} geometrically, one can represent a linear transformation T for a given matrix A, invert the linear transformation as T^{-1} and construct the matrix transformation A^{-1} for T^{-1}.

Example

Find A^{-1}, where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$