Theorem

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be eigenvectors corresponding to *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$ of an $n \times n$ matrix A. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

Corollary

If A is an $n \times n$ matrix with n *distinct* eigenvalues, then A is diagonalizable.

Example

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

An $n \times n$ matrix A may still be diagonalizable even if it does not have n distinct eigenvalues. Let λ_0 be an eigenvalue of A of multiplicity $m \leq n$. Let

$$E_{\lambda_0}(A) = \{\mathbf{x} \in \mathbf{R}^n : \; A\mathbf{x} = \lambda \mathbf{x}\}.$$

Then $E_{\lambda_0}(A)$ is a subspace of $\mathbf{R}^n$ called the **eigenspace** of A corresponding to the eigenvalue λ_0.

Theorem

Let λ_0 have algebraic multiplicity $m \leq n$, such that $c_A(\lambda) = (\lambda - \lambda_0)^m \tilde{c}_A(\lambda)$ and $\tilde{c}_A(\lambda_0) \neq 0$. Then $\dim(E_{\lambda_0}(A) \leq m$.

Corollary

The matrix A is diagonalizable if and only if $\dim(E_{\lambda_0}(A) = m$ for each eigenvalue λ_0 of algebraic mutliplicity m.

Example

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem

Let A be a symmetric matrix. Then A is diagonalizable and all eigenvalues are real.

Example

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$