Similar matrices

Two $n \times n$ matrices A and B are **similar**, written as $A \sim B$, if there is an invertible matrix P such that $B = P^{-1}AP$.

Example

An $n \times n$ matrix A is diagonalizable (or invertible) if and only if A is similar to a diagonal matrix (or to an identity matrix).

Properties of similarity

1. If $A \sim B$, then $B \sim A$.
2. If $A \sim B$ and $B \sim C$, then $A \sim C$.
3. If $A \sim B$, then $A^T \sim B^T$ and $A^k \sim B^k$ for all $k \geq 1$.
4. If $A \sim B$ and B is diagonalizable (or invertible), then A is also diagonalizable (or invertible).

Trace

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **trace** of A is the sum of its entries along the main diagonal:

$$\operatorname{tr}(A) := a_{11} + a_{22} + \cdots + a_{nn}.$$

Example

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Properties of $\operatorname{tr}(A)$

1. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.
2. $\operatorname{tr}(kA) = k \operatorname{tr}(A)$.
3. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Theorem

Let $A \sim B$. Then A and B have the same determinant, rank, trace, characteristic polynomial and eigenvalues.

Example

Matrices may have identical properties but still may not be self-similar.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem

An $n \times n$ matrix A is diagonalizable if and only if $\mathbf{R}^n$ has a basis of n eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of A. Then the matrix $P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$ is invertible and

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, $k = 1, ..., n$.

Example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$