Linear independence: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is called **linearly independent** if the linear combination,

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0},$$

has the only solution $t_1 = ... = t_k = 0$. Otherwise, the set is called **linearly dependent**.

Theorem: If the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, then any vector $\mathbf{u} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ has a unique representation as a linear combination of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Example:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix}$$

Theorem: Let V be a $n \times k$ matrix of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and X be a $k \times 1$ column matrix. The homogeneous system $AX = 0$ has a unique zero solution $X = 0$ (that is $\text{rank}(V) = k$) if and only if the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

Example:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 4 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 5 \\ -3 \\ 7 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 8 \\ -1 \\ 6 \end{pmatrix}$$

Basis and dimension

Let $U = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbf{R}^n$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is called a **basis** of U if these vectors are linearly independent. The **dimension** of a subspace U is the number of vectors in a basis of U.

Theorem: If U has two bases, e.g. $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, then $k = m$.

Examples:

What is the dimension of $\mathbf{R}^n$?

What is the dimension of $U = \{\mathbf{0}\}$?

Examples:

Let $U \subset \mathbf{R}^n$ and $\dim(U) = 1$. Show that U is a set of lines through the origin.

Let $U \subset \mathbf{R}^3$ and $\dim(U) = 2$. Show that U is a set of planes that contain the origin.

Prove that two vectors in $\mathbf{R}^n$ are linearly dependent if and only if they are parallel to each other.