Scalar product

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors and let θ be the angle between the two vectors measured in the same plane $(0 \leq \theta \leq \pi)$. The **scalar product** (or **dot product**) between the two vectors is

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos\theta.$$

Example: Simplify $\mathbf{v}_1 \cdot \mathbf{v}_2$ when (i) $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, (ii) $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, and (iii) $\mathbf{v}_1$ and $\mathbf{v}_2$ are anti-parallel.

Geometric properties:

- Let $\mathbf{v}$ be a vector with components (x, y, z). The length of the vector is
$$\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}.$$

- Let θ be an angle between two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then
$$\|\mathbf{v}_1 - \mathbf{v}_2\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 - 2\|\mathbf{v}_1\|\|\mathbf{v}_2\|\cos\theta$$

Example: Compute the distance between two points: $P_1(4, 3, 0)$ and $P_2(-1, 1, 3)$.

Theorem: Let $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. Then

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1^T \mathbf{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

Example: Compute the dot product between the vectors $\mathbf{v}_1 = \begin{pmatrix} -2 & 3 & 1 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}^T$.

Angle between two vectors: Given $\mathbf{v}_1$ and $\mathbf{v}_2$, one can find the angle θ between the two vectors:

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|}.$$

Example: Compute the angle (measured in radians) between the vectors $\mathbf{v}_1 = \begin{pmatrix} -2 & 3 & 1 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}^T$.

Properties:

- If $\mathbf{v}_1 \cdot \mathbf{v}_2 > 0$, then θ is acute $(0 \le \theta < \frac{\pi}{2})$
- If $\mathbf{v}_1 \cdot \mathbf{v}_2 < 0$, then θ is obtuse $(\frac{\pi}{2} < \theta \le \pi)$
- If $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, then $\theta = \frac{\pi}{2}$ ($\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal vectors)

Properties of the dot product:

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{u}$.
3. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 \geq 0$.
4. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$
5. $\mathbf{u} \cdot (\mathbf{v} \pm \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \pm \mathbf{u} \cdot \mathbf{w}$.

Example:

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)$$

Example: Show that the triangle with vertices $A(3, -2, 1)$, $B(5, 7, 0)$ and $C(-2, 1, 2)$ is not a right-angled triangle.