Linear Dynamical Systems

A **dynamical system** is defined by a set of finitely many variables that depend on time: the **state of the variables** at a **future time** is defined in terms of the state of the variables at the **present time**.

Example

Two competing television channels, called Channel 1 and Channel 2, have market shares $v_1(t)$ and $v_2(t)$. Suppose that each of the channels has initially a 50% market share. Assume also that over each one-year period, Channel 1 captures 10% of Channel 2's share and Channel 2 captures 20% of Channel 1's share. What is each channel's market share after one year? After k years?

General discrete linear dynamical system

Consider the time evolution of the discrete linear dynamical system:

$$V_{k+1} = AV_k, \qquad k \geq 0,$$

where A is a square $n \times n$ matrix and V_k is a column $n \times 1$ matrix.

Solution of the linear system:

$$V_k = A^k V_0, \qquad k \geq 1.$$

Long-time behavior is defined in the limit:

$$\lim_{k \to \infty} V_k = \lim k \to \infty A^k V_0 =?$$

Example

$$n = 1 : \quad A = a, \quad V_k = a^k V_0$$

Solution by diagonalization

Assume that A is a diagonalizable matrix, such that there exists a matrix of basic eigenvectors $P = [X_1, ..., X_n]$ and

$$P^{-1}AP = D = \text{diag}(\lambda_1, ...\lambda_n)$$

Let $V_k = PW_k$ such that

$$W_{k+1} = DW_k, \qquad k \geq 0$$

Let $W_0 = (c_1, ..., c_n)^T$. Then,

$$V_k = PD^kW_0 = c_1\lambda_1^kX_1 + ... + c_n\lambda_n^kX_n$$

Example

$$V_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = 0.7, \quad X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Convergence as $k \to \infty$

The eigenvalue with largest absolute value is called the **dominant** eigenvalue. As $k \to \infty$, the vector V_k is proportional to the eigenvector corresponding to the dominant eigenvalue.

Applications of discrete dynamical systems

- Markov chains for discrete probabilities
- nuclear and chemical reactions
- linear recurrences in numerical methods
- population dynamics

Example: Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k, \qquad k \geq 0,$$

such that $x_0 = x_1 = 1$.

Let $x_{k+1} = y_k$ and

$$V_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

such that $V_{k+1} = AV_k$.