Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and consider the linear homogeneous system:

$$AX = \lambda X.$$

For each *nonzero* column matrix $X \neq O$, a solution X is called an **eigenvector** of A and the number λ is called an **eigenvalue** of A.

Example

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 5 \end{bmatrix}$$

General procedure of computations of λ and X

1. Consider the homogeneous linear system

$$(\lambda I_n - A) \cdot X = 0.$$

2. Find values of λ when the determinant of $\lambda I_n - A$ is zero:

$$c_A(\lambda) = \det(\lambda I - A) = 0,$$

where $c_A(\lambda)$ is called the **characteristic** polynomial for the matrix A.

3. For each λ from the root of $c_A(\lambda) = 0$, find the most general *nonzero* solution X of the homogeneous system above. The most general solution can be expressed in the parameter form:

$$X = t_1 X_1 + ... + t_k X_k,$$

where $(t_1, ..., t_k)$ are independent parameters and $(X_1, ..., X_k)$ are basic eigenvectors.

Examples

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Theorem: There exist exactly n eigenvalues for an $n \times n$ matrix A. Eigenvalues of A are the same as eigenvalues of A^T.

Remarks:

1. If there are multiple eigenvalues, the number of basic eigenvectors can be smaller than the multiplicity of the eigenvalue.

2. Eigenvalues and eigenvectors can be complex-valued (that is they can be expressed by complex numbers).

Examples

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

where θ is real parameter.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$