Adjoints

Let $A = [a_{ij}]$ be an $n \times n$ matrix. We recall the Laplace expansion along the row i:

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik}(A),$$

where $C_{ik}(A)$ is a cofactor of a_{ik}. Let $C(A)$ be a **cofactor matrix** of A:

$$C(A) := [C_{ij}(A)].$$

The **adjoint** of A is the $n \times n$ matrix

$$\operatorname{adj}(A) := C(A)^T.$$

Example

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Example

$$A = \begin{bmatrix} -3 & 2 & 4 \\ 0 & 1 & 0 \\ -2 & 3 & 0 \end{bmatrix}$$

Theorem: Let A be a square $n \times n$ matrix and $\mathrm{adj}(A)$ be its adjoint. Then

$$A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = \det(A) \cdot I.$$

If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A).$$

<u>Example</u>

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Consider the linear square system,

$$A \cdot X = B,$$

where A is a $n \times n$ coefficient matrix, X and B are $n \times 1$ matrices. Let A be an invertible $n \times n$ matrix. Then,

$$X = A^{-1}B = \frac{1}{\det(A)}\text{adj}(A) \cdot B.$$

<u>Cramer's Rule</u>

If A is invertible $n \times n$ matrix, the system $A \cdot X = B$ has a unique solution $X = [x_1 \ \cdots \ x_n]^T$ given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \ \ldots, \ x_n = \frac{\det(A_n)}{\det(A)},$$

where the matrix A_j for each $j = 1, \ldots, n$ is obtained from A by replacing column j by B.

Example

$$
\begin{aligned}
-3x_1 + 2x_2 + 4x_3 &= 1 \\
x_2 &= 2 \\
-2x_1 + 3x_2 &= 3
\end{aligned}
$$