The idea of the determinant

A scalar (number) is associated to a square matrix A, which is called the **determinant** of A and denoted as $\det(A)$. The zero or non-zero values of the scalar are associated to existence or non-existence of the matrix inverse A^{-1}.

Examples

$$n = 1, \quad A = [a], \quad A^{-1} = \frac{1}{a}, \quad \det(A) := a.$$

$$n = 2,\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},\ \det(A) := ad - bc.$$

- If $\det(A) = 0$, the matrix A is singular (no A^{-1} exists).
- If $\det(A) \neq 0$, the matrix A is invertible (A^{-1} exists).

<u>Example</u>

$$A = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

Laplace expansion

The determinant of an $n \times n$ matrix is computed from the determinant of an $(n-1) \times (n-1)$ matrix by using the **Laplace expansion formula**.

Let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from the $n \times n$ matrix A. The (i, j)**-cofactor** of A is the number

$$C_{ij}(A) := (-1)^{i+j} \det(A_{ij}).$$

Example

$$A = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 1 & 7 \\ 9 & -2 & -6 \end{bmatrix}$$

Theorem: Let A be an $n \times n$ matrix. The Laplace expansion of the determinant of A along row i is given by

$$\det(A) = a_{i1}C_{i1}(A) + a_{i2}C_{i2}(A) + \cdots + a_{in}C_{in}(A).$$

The result is invariant with respect to the selected row i.

Example

$$A = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 1 & 7 \\ 9 & -2 & -6 \end{bmatrix}$$

Laplace expansion becomes shorter if some entries in the row are zeros.

Instead of expansion along the row i, one can use the Laplace expansion along the column j:

$$\det(A) = a_{1j}C_{1j}(A) + a_{2j}C_{2j}(A) + \cdots + a_{nj}C_{nj}(A).$$

The result is invariant with respect to the selected column j.

<u>Example</u>

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 2 & -1 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 5 & -2 & -2 \end{bmatrix}.$$