Example

The linear system of 3 equations for 3 unknowns:

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

The **augmented matrix** of the linear system:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}.$$

We shall solve the linear system by performing **elementary row operations** on *rows* of the augmented matrix:

I. Interchange two rows.

II. Multiply one row by a nonzero number.

III. Add a multiple of one row to another row.

Theorem: Elementary row operations do not change solutions of the linear system.

After a sequence of elementary row operations, the augmented matrix is reduced to the **row-echelon form**:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Properties of the row-echelon form:

1. All zero rows are at the bottom.
2. The first nonzero entry in each row is 1.
3. Each leading 1 is to the right of all leading 1's in the rows above it.

Applying a few more elementary row operations, we can reduce the augmented matrix to the **reduced row-echelon form**:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3. \end{bmatrix}$$

Properties of the reduced row-echelon form:

1-3. Properties of the row-echelon form.

4. Each leading 1 is the only nonzero entry in its column.

The procedure of reduction of the augmented matrix to reduced row-echelon form is called **Gaussian elimination**.

A system of equations is called **consistent** if it has a solution. It is called **inconsistent** if it has no solutions at all.

Complete characterization of solutions of the consistent systems can be developed from the reduced row-echelon form of the augmented matrix. A general result is that any matrix can be brought to the reduced row-echelon form.