Justification of a nonlinear Schrödinger model for polymers

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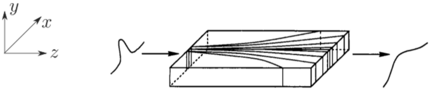
Photopolymers

Photopolymers

- experience permanent change in refractive index under illumination due to triggered polymerization process;
- demonstrate wave guide trapping, mode evolution and other interesting nonlinear effects observed experimentally [Villafranca, Saravanamuttu, 2008];
- are modelled with Maxwell's and NLS equations [Monro et al, 2001].

Model for pulse propagation



Wave-Maxwell system:

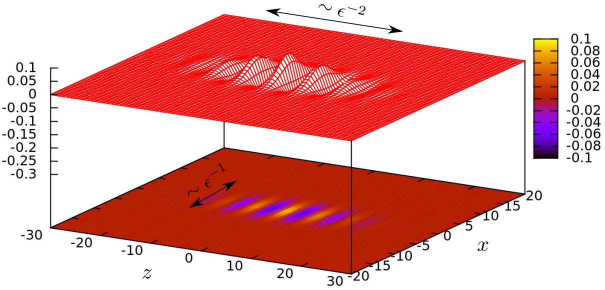
$$\begin{cases} \partial_x^2 E + \partial_z^2 E - (1+m)\,\partial_t^2 E = 0, \\ \frac{\partial m}{\partial t} = E^2. \end{cases} \qquad (x,z) \in \mathbb{R}^2, \;\; t \in \mathbb{R}_+$$

Initial conditions:

$$E|_{t=0} =: E_0 = \epsilon^{\frac{s+2}{2}} A_0\left(\epsilon x, \epsilon^2 z\right) e^{i\omega_0 z} + c.c., \quad s \geq 2, \quad 0 < \epsilon \ll 1,$$
$$\partial_t E|_{t=0} =: E_1 = -i\omega_0 \epsilon^{\frac{s+2}{2}} A_0\left(\epsilon x, \epsilon^2 z\right) e^{i\omega_0 z} + c.c., \quad m|_{t=0} = 0.$$

Model for pulse propagation

Initial pulse:



NLS model:

$$\begin{cases} \partial_X^2 A + 2i\omega_0 \left(\partial_Z A + \partial_T A\right) + \omega_0^2 m_0 A = 0, \\ \partial_T m_0 = 2\left|A\right|^2. \end{cases} \qquad (X,Z) \in \mathbb{R}^2, \ \ T \in \mathbb{R}_+$$

Initial conditions: $A|_{T=0} =: A_0(X,Z)$, $m_0|_{T=0} = 0$.

Reduction to a NLS model

Asymptotic expansion:

$$\begin{cases} E(x,z,t) = \epsilon^{\frac{s+2}{2}} \underbrace{\left(A(X,Z,T)\, e^{i\omega_0 (z-t)} + c.c. \right)}_{=:(A)_{\omega_0}} + U(x,z,t), \\ m(x,z,t) = \epsilon^2 m_0 (X,Z,T) + N(x,z,t), \end{cases}$$

where $\quad X := \epsilon x, \quad Z := \epsilon^2 z, \quad T := \epsilon^s t, \quad s \geq 2.$

Choosing $s = 2$, leading order approximation yields the NLS model stated above; for $s > 2$, the NLS equation does not contain the $\partial_T A$ term.

Our goal is to show that the residual terms $U(x,z,t)$ and $N(x,z,t)$ are indeed small !

Main result

Theorem

Given initial data $A_0 \in H^8\left(\mathbb{R}^2\right)$, let A, m_0 be local solutions to the NLS system for $T \in [0, T_0]$ with sufficiently small $T_0 > 0$. There exist $\epsilon_0 > 0$ and a unique solution E, m of the wave-Maxwell system for $t \in \left[0, T_0/\epsilon^2\right]$ such that, for any $\epsilon \in (0, \epsilon_0)$,

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|E - \epsilon^2 (A)_{\omega_0}\right\|_{H^3(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right),$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|m - \epsilon^2 m_0\right\|_{H^2(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right).$$

Idea of the proof

We obtain

- local well-posedness of the wave-Maxwell system (Kato's theory for symmetric hyperbolic systems);
- continuation arguments for the local solution (energy method);
- local well-posedness of the NLS system (Banach fixed-point theorem);
- residual equations in suitable for analysis form (near-identity transformations);
- control of error terms (a priori energy estimates).

Wave-Maxwell system: local well-posedness

The wave-Maxwell system can be rewritten in the form

$$\begin{cases} \partial_t \mathbf{v} + A_1(\mathbf{v})\,\partial_x \mathbf{v} + A_2(\mathbf{v})\,\partial_z \mathbf{v} = \mathbf{f}(\mathbf{v}), \\ \mathbf{v}\big|_{t=0} = \mathbf{v}_0 \end{cases}$$

where $\mathbf{v} := \left(\partial_t E,\, \frac{\partial_x E}{(1+m)^{1/2}},\, \frac{\partial_z E}{(1+m)^{1/2}},\, E,\, \partial_x m,\, \partial_z m,\, m\right)^T$, $\mathbf{v}_0 := (E_1,\, \partial_x E_0,\, \partial_z E_0,\, E_0,\, 0,\, 0,\, 0)^T$, and A_1, A_2 are symmetric matrices.

Then, Kato's theory for symmetric quasi-linear hyperbolic systems gives, for some $t_0 > 0$,

$$\mathbf{v} \in C\left([0, t_0], H^s\left(\mathbb{R}^2\right)\right) \cap C^1\left([0, t_0], H^{s-1}\left(\mathbb{R}^2\right)\right)$$

providing $\mathbf{v}_0 \in H^s\left(\mathbb{R}^2\right)$.

Wave-Maxwell system: local well-posedness

Proposition

For any integer $s \geq 3$, the unique local solution of the wave-Maxwell system exists in the space

$$E \in C\left([0,t_0], H^{s+1}\left(\mathbb{R}^2\right)\right) \cap C^1\left([0,t_0], H^{s}\left(\mathbb{R}^2\right)\right) \cap C^2\left([0,t_0], H^{s-1}\left(\mathbb{R}^2\right)\right)$$

$$m \in C^1\left([0,t_0], H^{s+1}\left(\mathbb{R}^2\right)\right) \cap C^2\left([0,t_0], H^{s}\left(\mathbb{R}^2\right)\right) \cap C^3\left([0,t_0], H^{s-1}\left(\mathbb{R}^2\right)\right)$$

and depends continuously on the initial data $E_0,\ E_1 \in H^s\left(\mathbb{R}^2\right)$.

Wave-Maxwell system: local well-posedness

Theorem (continuation of local solution)

Local solution of the wave-Maxwell system does not blow up in H^4-norm as $t \to t_0$ if

$$\sup_{t\in[0,t_0]} \left(\|E\|_{L^\infty(\mathbb{R}^2)} + \|\partial_t E\|_{L^\infty(\mathbb{R}^2)} + \|\nabla E\|_{L^\infty(\mathbb{R}^2)} \right) < \infty,$$

and the initial data norms $\|E_0\|_{H^3(\mathbb{R}^2)}$ and $\|E_1\|_{H^2(\mathbb{R}^2)}$ are small enough.

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We prove this by energy method. The derivatives $\partial^\alpha E$, $|\alpha| \leq 4$, can be controlled in L^2-norm by constructing four energy functionals which are consequently shown to be bounded on times $t \in [0, t_0]$.

NLS system: local well-posedness

Theorem

For any integer $s \geq 2$ and $\delta > 2 \sup_{T \in \mathbb{R}_+} \|A_0\|_{H^s(\mathbb{R}^2)}$, there exist a positive T_0 and a unique solution

$$A \in C\left([0, T_0], H^s\left(\mathbb{R}^2\right)\right) \cap C^1\left([0, T_0], H^{s-2}\left(\mathbb{R}^2\right)\right)$$

to the NLS system with $A|_{T=0} = A_0$ and $\sup_{T \in [0, T_0]} \|A\|_{H^s(\mathbb{R}^2)} \leq \delta$.

NLS system: local well-posedness

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Proof is application of the Banach fixed-point theorem to

$$A(X, Z, T) = S_T(X) \star A_0(X, Z - T)$$
$$+ i \int_0^T S_{T-\tau}(X) \star [m_0(X, Z - T + \tau, \tau) A(X, Z - T + \tau, \tau)] \, d\tau,$$

where $S_T(X) := \frac{1}{\sqrt{4\pi T}} e^{-\frac{i\pi}{4}} e^{\frac{iX^2}{4T}}$.

Residual equations

Recall

$$\begin{cases} E(x,z,t) = \epsilon^2 (A)_{\omega_0} + U(x,z,t), \\ m(x,z,t) = \epsilon^2 m_0 + N(x,z,t). \end{cases}$$

Residual terms solve the equations

$$\begin{cases} \partial_x^2 U + \partial_z^2 U - \left(1 + \epsilon^2 m_0 + N\right) \partial_t^2 U = -\epsilon^2 \left(R_2^{(U)}\right)_{\omega_0} N - \epsilon^6 \left(R_6^{(U)}\right)_{\omega_0}, \\ \partial_t N = \epsilon^4 \left(A^2\right)_{2\omega_0} + 2\epsilon^2 (A)_{\omega_0} U + U^2 \end{cases}$$

with

$$U|_{t=0} = \partial_t U|_{t=0} = 0, \quad N|_{t=0} = 0,$$

where
$$R_2^{(U)} := \omega_0^2 A + 2i\omega_0 \epsilon^2 \partial_T A - \epsilon^4 \partial_T^2 A,$$
$$R_6^{(U)} := \partial_Z^2 A - \left(1 + \epsilon^2 m_0\right) \partial_T^2 A + 2i\omega_0 m_0 \partial_T A.$$

Residual equations

The normal form transformation

$$V = U - \epsilon^4 (B)_{\omega_0} - \epsilon^4 (D)_{3\omega_0},$$

$$M = N - \epsilon^4 N_0 + 2\epsilon^2 \left(\frac{A}{i\omega_0}\right)_{\omega_0} V + \epsilon^4 \left(\frac{A^2}{2i\omega_0}\right)_{2\omega_0} + \epsilon^6 R_6^{(M)}$$

allows us to work with

$$\begin{cases} \partial_x^2 V + \partial_z^2 V - \left(1 + \epsilon^2 m_0 + N\right) \partial_t^2 V = 2i\omega_0 \epsilon^4 |A|^2 V - \epsilon^2 \omega_0^2 (A)_{\omega_0} M \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \epsilon^8 R_8^{(V)}, \\ \partial_t M = \epsilon^8 R_8^{(M)} + 2\epsilon^4 R_4^{(M)} V + 2\epsilon^2 \left(\frac{A}{i\omega_0}\right)_{\omega_0} \partial_t V + V^2. \end{cases}$$

Control of error terms

The following quantities can be controlled, for $t \in \left[0,\, T_0/\epsilon^2\right]$,

$$\mathcal{H}_1 := \frac{1}{2}\int_{\mathbb{R}^2} \left[\left(1+\epsilon^2 m_0 + N\right) V_t^2 + V_x^2 + V_z^2\right] dxdz,$$

$$\begin{aligned}\mathcal{H}_2 \quad &:= \quad \frac{1}{2}\int_{\mathbb{R}^2} \left[\left(1+\epsilon^2 m_0 + N\right) V_{tt}^2 + \left(2+\epsilon^2 m_0 + N\right)\left(V_{xt}^2 + V_{zt}^2\right)\right. \\ &+ \quad \left. V_{xx}^2 + V_{zz}^2 + 2V_{xz}^2\right] dxdz,\end{aligned}$$

$$\begin{aligned}\mathcal{H}_3 \quad &:= \quad \frac{1}{2}\int_{\mathbb{R}^2} \left[\left(1+\epsilon^2 m_0 + N\right)\left(V_{xxt}^2 + V_{zzt}^2 + V_{ttt}^2\right) + \frac{1}{2}\left(V_{xxx}^2 + V_{zzz}^2\right)\right. \\ &+ \quad \left. V_{xxz}^2 + V_{xzz}^2 + \frac{1}{2}\left(V_{xxx} - 2N_x V_{tt}\right)^2 + \frac{1}{2}\left(V_{zzz} - 2N_z V_{tt}\right)^2\right] dxdz.\end{aligned}$$

Control of error terms

For example, multiplying the first equation by $\partial_t U$ and integrating in x, z over $\mathbb{R}^2$:

$$\begin{aligned}\frac{d\mathcal{H}_1}{dt} &\leq 2\epsilon^4 \|A\|_{L^\infty}^2 \mathcal{H}_1 + \|N_t\|_{L^\infty} \mathcal{H}_1 + 2\sqrt{2}\epsilon^2\omega_0^2 \|A\|_{L^\infty} \|M\|_{L^2} \mathcal{H}_1^{1/2} \\ &+ 2\sqrt{2}\epsilon^4\omega_0 \|A\|_{L^\infty}^2 \|V\|_{L^2} \mathcal{H}_1^{1/2} + \sqrt{2}\epsilon^{13/2} \left\| R_8^{(V)} \right\|_{L^2_{x,z}} \mathcal{H}_1^{1/2}.\end{aligned}$$

Using the second equation and the estimates

$$\|\cdot\|_{L^2} \leq T_0\epsilon^{-2} \sup_{T\in[0,T_0]} \|\partial_t(\cdot)\|_{L^2}, \quad \text{for } t \in \left[0, T_0/\epsilon^2\right],$$

$$\frac{d\left(\mathcal{H}_1^{1/2}\right)}{dt} \leq I_1 \mathcal{H}_1^{1/2} + J_1 \quad \Rightarrow \quad \mathcal{H}_1^{1/2}(t) \leq T_0\epsilon^{-2} J_1 e^{I_1 T_0 \epsilon^{-2}}.$$

Control of error terms

Eventually, we obtain the bounds

$$\sup_{t\in[0,T_0/\epsilon^2]} \|V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right), \quad \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla V\|_{L^2} = \mathcal{O}\left(\epsilon^{9/2}\right),$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|\Delta V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right), \quad \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla\Delta V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right),$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|M\|_{L^2} = \mathcal{O}\left(\epsilon^{9/2}\right), \quad \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla M\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right),$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|\Delta M\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right).$$

Numerical illustrations in 1D

1D set-up:

$$\begin{cases} \partial_z^2 U - \left(1 + \epsilon^2 m_0 + N\right) \partial_t^2 U = -\epsilon^2 \left(R_2^{(U)}\right)_{\omega_0} N - \epsilon^6 \left(R_6^{(U)}\right)_{\omega_0}, \\ \partial_t N = \epsilon^4 \left(A^2\right)_{2\omega_0} + 2\epsilon^2 (A)_{\omega_0} U + U^2. \end{cases}$$

The choice of initial data $A_0(Z) = \text{sech}(Z)$ leads to

$$A(Z,T) = \text{sech}(Z-T) \exp\left[i\omega_0 \left(\log\left(\frac{\cosh Z}{\cosh(Z-T)}\right) - T\tanh(Z-T)\right)\right],$$
$$m_0(Z,T) = 2\left(\tanh Z - \tanh(Z-T)\right).$$

Numerical illustrations in 1D

Rewriting

$$\begin{cases} \partial_t^2 U = \frac{1}{a+N}\partial_z^2 U - \frac{bN+c}{a+N}, \\ \partial_t N = U^2 + dU + e, \end{cases}$$

we use explicit 2-step Adams-Bashforth and Crank-Nicholson methods

$$N_i^{(j+1)} = N_i^{(j)} + \frac{3}{2}\Delta t \left[\left(U_i^{(j)}\right)^2 + d_i^{(j)} U_i^{(j)} + e_i^{(j)} \right] - \frac{1}{2}\Delta t \left[\left(U_i^{(j-1)}\right)^2 + d_i^{(j-1)} U_i^{(j-1)} + e_i^{(j-1)} \right],$$

$$\begin{aligned} \frac{U_i^{(j+1)} - 2U_i^{(j)} + U_i^{(j-1)}}{(\Delta t)^2} = \frac{1}{2} \Bigg[& \frac{1}{a_i^{(j+1)} + N_i^{(j+1)}} \left(\frac{U_{i+1}^{(j+1)} - 2U_i^{(j+1)} + U_{i-1}^{(j+1)}}{(\Delta z)^2} - b_i^{(j+1)} N_i^{(j+1)} - c_i^{(j+1)} \right) \\ & + \frac{1}{a_i^{(j-1)} + N_i^{(j-1)}} \left(\frac{U_{i+1}^{(j-1)} - 2U_i^{(j-1)} + U_{i-1}^{(j-1)}}{(\Delta z)^2} - b_i^{(j-1)} N_i^{(j-1)} - c_i^{(j-1)} \right) \Bigg]. \end{aligned}$$

Numerical illustrations in 1D

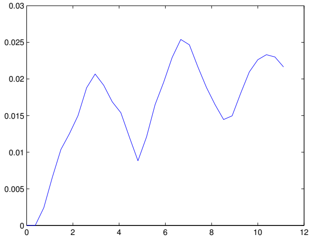

Figure: $\|U\|_{H^3(\mathbb{R})}$ versus time t

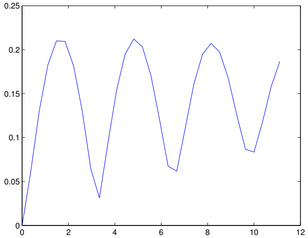

Figure: $\|N\|_{H^2(\mathbb{R})}$ versus time t

- Solutions for $\epsilon = 0.3$ animated: U, N.
- Estimates:

$$\sup_{t\in[0,1]} \|U\|_{H^3(\mathbb{R})} = \mathcal{O}\left(\epsilon^{\hat{\alpha}}\right), \quad \sup_{t\in[0,1]} \|N\|_{H^2(\mathbb{R})} = \mathcal{O}\left(\epsilon^{\hat{\beta}}\right),$$

where $\hat{\alpha} = 4.9972$, $\hat{\beta} = 3.0028$.

The End

Thank You for the attention !!!