STUDY OF VORTICES IN TWO-DIMENSIONAL HARMONIC POTENTIALS

Math-790

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Introduction

- What is Bose-Einstein condensation (BEC)?

•The nonlinear evolution equation, called the Gross-Pitaevskii equation, models BEC in the mean-field approximation.

$$i\epsilon u_t + \epsilon^2(u_{xx} + u_{yy}) + (1 - x^2 - y^2 - |u|^2)u = 0.$$

where ϵ is a small parameter inversely proportional to chemical potential and $u(x, y, t) \in \mathbb{C}$ is the wave function.

- This project consists of following parts:
  - Study of vortex solutions near the bifurcation point.
  - Existence of a vortex solution.
  - Uniqueness of the positive vortex solution.
  - Numerical result.

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Local Bifurcation of Vortex Solutions

• We define the Schrödinger operator $\mathcal{H}_0$ for a two-dimensional harmonic oscillator in the form:

$$\mathcal{H}_0 = -\epsilon^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + x^2 + y^2 - 1, \quad \epsilon > 0.$$

where the domain of $\mathcal{H}_0$ is:

$$Dom(\mathcal{H}_0) = \left\{ f \in H^2(\mathbb{R}^2) : |x|^2 f \in L^2(\mathbb{R}^2) \right\}.$$

• The stationary Gross-Pitaevskii equation can be written in the form:

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- To find the spectrum of $\mathcal{H}_0$ we write the eigenvalue equation:

$$\mathcal{H}_0 f = \lambda f, \quad f \in Dom(\mathcal{H}_0),$$

where λ stands for the eigenvalues and f for the corresponding eigenfunctions.

- Using the separation of variables to represent the wave function in the product form $f(x,y) = \varphi(x)\psi(y)$ and the properties of harmonic oscillators we can derive the eigenvalues of $\mathcal{H}_0$ as:

$$\sigma(\mathcal{H}_0) = \left\{\lambda_{k,m}(\epsilon) = -1 + 2\epsilon(k+m+1), \, (k,m) \in \mathbb{N}_0^2\right\}.$$

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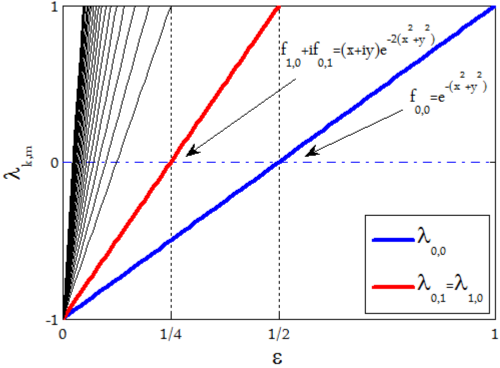

Figure 1: Eigenvalues $\lambda_{k,m}$ vs ϵ.

Local Bifurcation of Vortex Solutions

• We use the method of Lyapunov-Schmidt Reduction to study the local bifurcation:

Theorem 1

Let $\mu = \frac{1}{16} - \epsilon^2$. There is $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, there exist vortex solutions of the form,

$$u = \sqrt{128\mu}(x \pm iy)e^{-2(x^2+y^2)} + \mathcal{O}_{H^2(\mathbb{R}^2)}(\sqrt{\mu^3}),$$

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Existence of Vortex Solution

• We use calculus of variations to prove the existence of the vortex solution $u(x, y) = \phi(r)e^{i\theta}$ in polar coordinate (r, θ) on a truncated interval $[0, R] \ni r$.

• The associated energy functional is:

$$E_1(v) = \int_0^\infty \left[\underbrace{\epsilon^2 \left(\frac{dv}{dr}\right)^2 + \frac{\epsilon^2}{r^2} v^2 + (r^2 - 1)v^2}_{} + \frac{1}{2} v^4 \right] rdr.$$

• The stationary G-P equation is the Euler-Lagrange equation for $E_1(v)$:

$$\epsilon^2 \left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi \right) + (1 - r^2 - \phi^2)\phi = 0.$$

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Existence of Vortex Solution

Theorem 2

For all $0 < \epsilon < \frac{1}{4}$, the energy functional $E_1(v)$, has a nonzero global minimizer ϕ in the energy space:

$$X_R = \{v \in H_r^1(0, R) : rv \in L_r^2(0, R)\}.$$

- Remark: To show that the energy functional is bounded we truncated $\mathbb{R}_+$ on a bounded interval $[0, R]$, subjected to the Dirichlet boundary condition $v|_{r=R} = 0$.

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For any fixed $0 < \epsilon < \frac{1}{4}$, the positive vortex solution ϕ, is unique.

- Proof by ODE technique.

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Numerical Approximation of The Vortex Solution.

- Shooting Method has been used to derive an approximation for the positive vortex solution for any fixed ϵ on the interval $[0, 1]$.

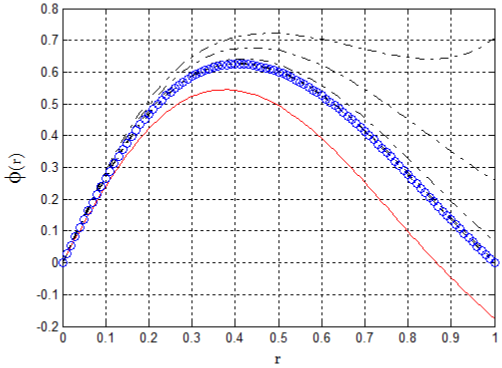

Figure 2: Vortex solution for $\epsilon = \frac{1}{6}$.

Open Problems.

• L-S Reduction suggest bifurcation of other (dipole) solutions for $\epsilon < \frac{1}{4}$. Persistence of the dipole solutions was not studied.

• Existence of vortex solution was proven on the truncated interval $[0, R]$. Can we prove the existence without the compactness assumption?

• We proved uniqueness of positive vortex solutions. Can we prove that positive vortex solutions exist?

• The shooting method was developed on the interval $[0, 1]$ but it has bad accuracy. What numerical methods can be used to improve the accuracy of numerical approximation?

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