Characterization of elliptic solutions in the defocusing mKdV equation

Dmitry E. Pelinovsky
McMaster University (Canada)
http://dmpeli.math.mcmaster.ca

In collaboration with
Ana Mucalica (Vancouver), Mark Hoefer (Boulder),
Lynnyngs Kelly Arruda (Sao Paulo), Rudi Weikard (Alabama)

Section 1

Motivations and the state-of-art for the KdV equation:

- stability of the traveling wave background
- breathers on traveling wave background

Motivations

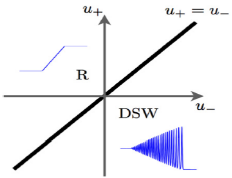
Dispersive hydrodynamics for the canonical model of the Korteweg–de Vries (KdV) equation has been well studied.

$$u_t + 6uu_x + u_{xxx} = 0,$$

with the step-like data

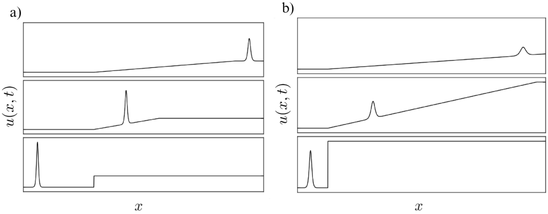
$$\lim_{x\to-\infty} u(t,x) = u_-, \qquad \lim_{x\to+\infty} u(t,x) = u_+.$$

The step-like initial data results in the appearance of a rarefaction wave (RW) if $u_+ > u_-$ and a dispersive shock wave (DSW) if $u_+ < u_-$.

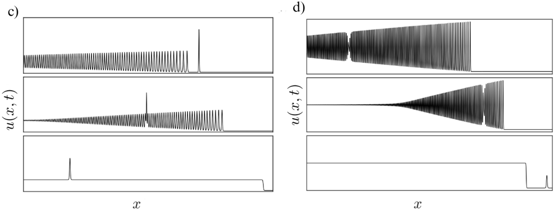


Motivations

Solitons interactions with RWs and DSWs:



Soliton-RW:
a) Tunneling.
b) Trapping.

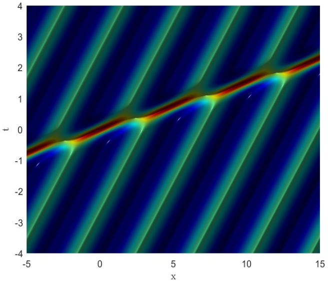


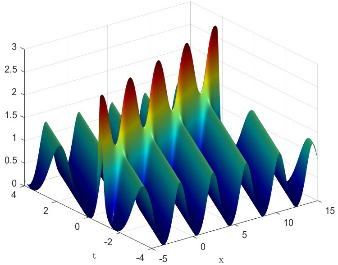
Soliton-DSW:
a) Tunneling.
b) Trapping.

M. J. Ablowitz, J. T. Cole, G. A. El, M. A. Hoefer, X. Luo, Stud. Appl. Math. (2023)

Motivations

Bright breathers on the modulationally stable TW of the KdV equation $u_t + 6uu_x + u_{xxx} = 0$





Move faster that the periodic wave and induce the phase shift.

E. Kuznetsov, A. Mikhailov, JETP (1974)

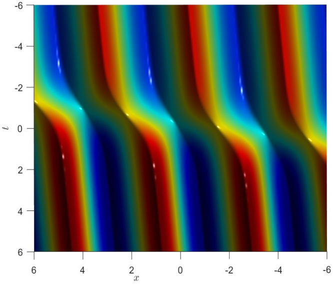
M. Hoefer, A. Mucalica, D.P., JPA (2023)

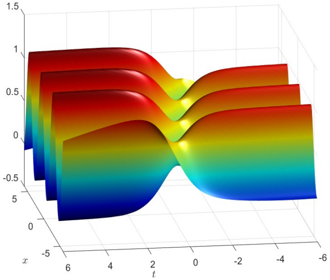
M. Bertola, R. Jenkins, A. Tovbis, Nonlinearity (2023)

Motivations

Dark breathers on the modulationally stable TW of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$





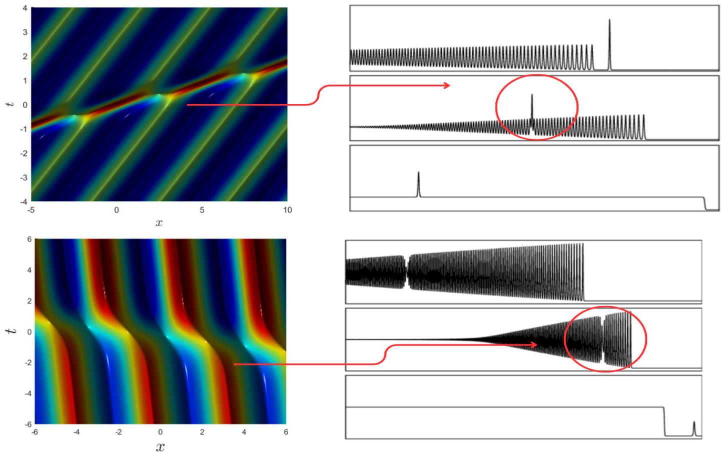
Move slower that the periodic wave and induce the phase shift.

E. Kuznetsov, A. Mikhailov, JETP (1974)

M. Hoefer, A. Mucalica, D.P., JPA (2023)

M. Bertola, R. Jenkins, A. Tovbis, Nonlinearity (2023)

Motivations



Further applications:

Y. Mao, S. Chandramouli, W. Xu, M. Hoefer, Phys. Rev. Lett. (2023)

State-of-art for construction of breathers

Traveling waves $u(x,t) = \phi(x - ct)$ of the KdV equation satisfy

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Rightarrow \quad \phi''' + 6\phi\phi' - c\phi' = 0.$$

After integration(s), it yields

$$\phi'' + 3\phi^2 - c\phi = b \quad \Rightarrow \quad (\phi')^2 + 2\phi^3 - c\phi^2 = 2b\phi + d$$

with three parameters (b, c, d). Due to scaling transformation

$$u(x,t) \mapsto \alpha u(\alpha x, \alpha^3 t)$$

and Galilean transformation

$$u(x,t) \mapsto \beta + u(x - 6\beta t, t),$$

only one parameter is independent.

State-of-art for construction of breathers

The normalized form of TW solutions is

$$\phi(x) = 2k^2 \mathrm{cn}^2(x;k), \quad b_0 = 4k^2(1-k^2), \quad c_0 = 4(2k^2-1), \quad d_0 = 0.$$

A general TW solution is obtained by the scaling and Galilean transformations.

sn, cn, and dn are real-valued Jacobi elliptic functions on $\mathbb{R}$ with

$$\mathrm{sn}^2(x;k) + \mathrm{cn}^2(x;k) = 1, \quad \mathrm{dn}^2(x;k) + k^2\mathrm{sn}^2(x;k) = 1,$$

with elliptic modulus $k \in (0,1)$ between $k \to 0$ (trigonometric functions) and $k \to 1$ (hyperbolic functions).

Elliptic functions sn^2, cn^2, and dn^2 are double-periodic in $\mathbb{C}$ with periods $2K(k)$ and $2iK'(k)$, where $K'(k) = K(\sqrt{1-k^2})$.

State-of-art for construction of breathers

Our method of construction is the Darboux–Backlund transformation

$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v),$$

where $u(x,t) = \phi(x-ct)$ is the TW solution of the KdV equation and $v(x,t) = w(x-ct)e^{\omega t}$ is a solution of the Lamé equation

$$w''(x) + 2k^2\mathrm{cn}^2(x;k)w(x) + \lambda w(x) = 0$$

with some uniquely determined $\omega = \omega(\lambda)$ from the linear equation

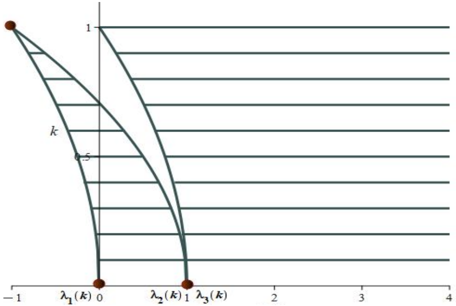
$$\omega w(x) - cw'(x) = -3\phi'(x)w(x) - 6\phi w'(x) - 4w'''(x).$$

The eigenfunction $v(x,t)$ is a solution of the linear Lax system

$$\mathcal{L}(u)v = \lambda v, \quad \frac{\partial v}{\partial t} = \mathcal{M}(u)v,$$

associated with the KdV equation.

State-of-art for construction of breathers



Bright breathers correspond to λ in semi-infinite gap.

Dark breathers correspond to λ in the finite gap.

Construction of breathers

The Lamé equation for a given $\lambda \in \mathbb{R}$

$$w''(x) + 2k^2 \mathrm{cn}^2(x;k) w(x) + \lambda w(x) = 0$$

is solved with the explicit functions

$$w_\pm(x) = \frac{H(x \pm \alpha)}{\Theta(x)} e^{\mp x Z(\alpha)}, \quad \lambda = 1 - 2k^2 + k^2 \mathrm{cn}(\alpha;k)$$

where $\alpha \in \mathbb{C}$ is a new parameter and Jacobi' theta functions are

$$H(x) = \theta_1\left(\frac{\pi x}{2K}\right) = 2\sum_{n=1}^{\infty}(-1)^{n-1} q^{(n-1/2)^2} \sin\left(\frac{\pi(2n-1)x}{2K}\right),$$

$$\Theta(x) = \theta_4\left(\frac{\pi x}{2K}\right) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos\left(\frac{\pi(2n)x}{2K}\right),$$

where $H(x) = \sqrt{k}\,\mathrm{sn}(x)\Theta(x)$.

D. F. Lawden, *Elliptic Functions and Applications*, (Springer, 2013)

Construction of breathers

The time evolution of the eigenfunctions follows from the separation of variables in the Lax system:

$$v_{\pm}(x,t) = \frac{H(x-ct \pm \alpha)}{\Theta(x-ct)} e^{\mp(x-ct)Z(\alpha) \mp t\omega(\alpha)},$$

where $\omega(\alpha)$ is found at $x = 0$:

$$\omega(\alpha) = 4(\lambda + k^2 - 1)\left[\frac{\Theta'(\alpha)}{\Theta(\alpha)} - \frac{H'(\alpha)}{H(\alpha)}\right].$$

Construction of breathers

Darboux transformation for λ in the semi-infinite gap is used with

$$v(x,t) = c_+ v_+(x,t) + c_- v_-(x,t)$$

where $v_\pm(x,t) > 0$ and $c_\pm > 0$. This yields the new solution for the bright breather:

$$\hat{u} = u + 2\frac{\partial^2}{\partial x^2}\log(v_0) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log\tau,$$

where the τ-function is

$$\tau = \Theta(x - c_0 t + \alpha_b)e^{\kappa_b(x - c_b t + x_0)} + \Theta(x - c_0 t - \alpha_b)e^{-\kappa_b(x - c_b t + x_0)}$$

with uniquely defined parameters $c_b > c_0$, $\kappa_b > 0$, and $\alpha_b \in [0, K]$.

Construction of breathers

Darboux transformation for λ in the finite gap is used with

$$v(x,t) = c_+ v_+(x,t) + c_- v_-(x,t)$$

but $v_\pm(x,t)$ are sign-indefinite. However, translation of the new solution $\hat{u} = \hat{u}(x + iK', t)$ yields a bounded solution for the dark breather:

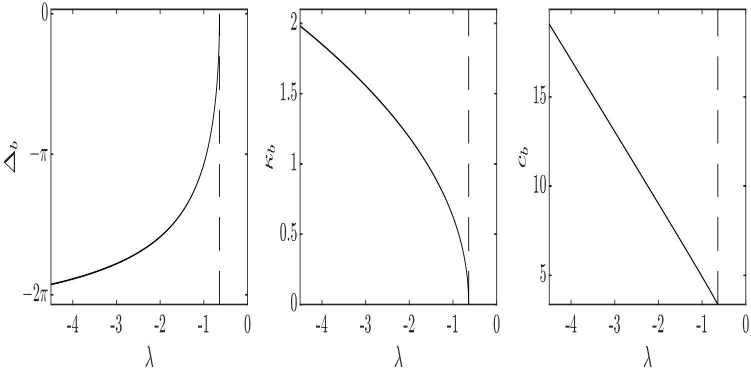
$$\hat{u} = u + 2\frac{\partial^2}{\partial x^2}\log(v_0) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log\tau,$$

where the τ-function is

$$\tau = \Theta(x - c_0 t + \alpha_d)e^{-\kappa_d(x - c_d t + x_0)} + \Theta(x - c_0 t - \alpha_d)e^{\kappa_d(x - c_d t + x_0)}$$

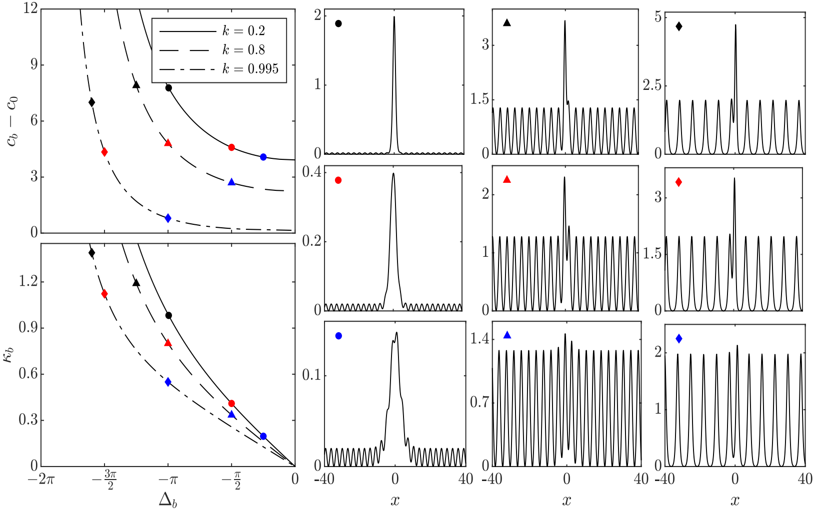
with uniquely defined parameters $c_d < c_0$, $\kappa_d > 0$, and $\alpha_d \in [0, K]$.

Bright breathers

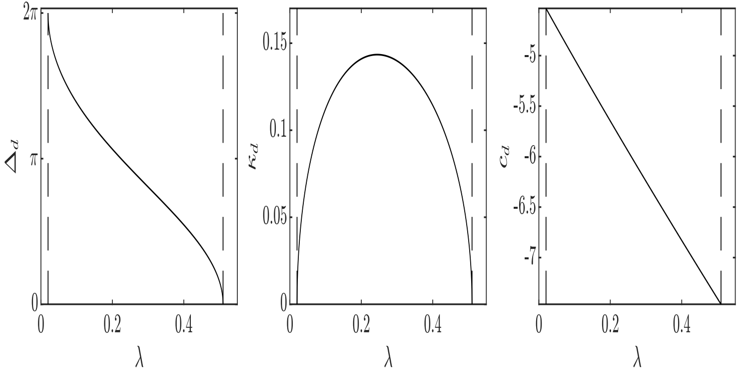


Here $\Delta_b = 2\pi\alpha_b/K(k)$ is normalized phase shift.
We can prove $\Delta_b'(\lambda) > 0$, $\kappa_b'(\lambda) < 0$, and $c_b > c_0$.

Bright breathers

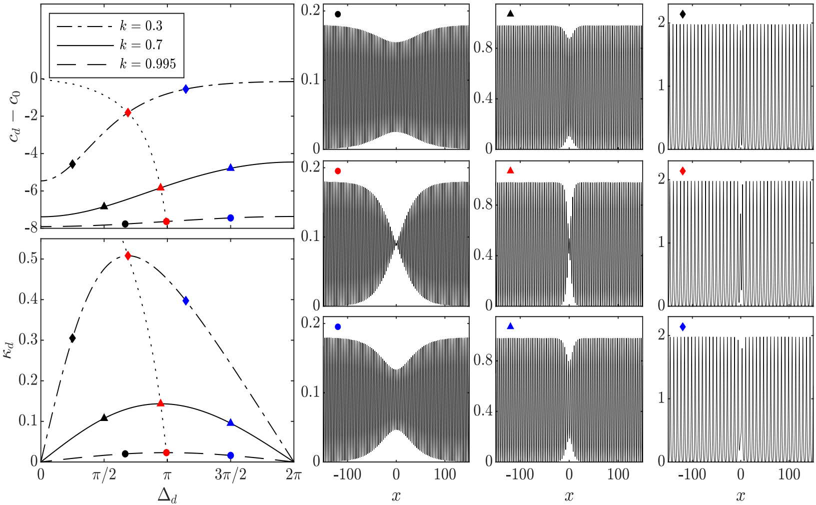


Dark breathers



Here $\Delta_d = 2\pi\alpha_d/K(k)$ is normalized phase shift.
We can prove $\Delta_d'(\lambda) < 0$, $\max \kappa_d(\lambda)$, and $c_d < c_0$.

Dark breathers



State-of-art of stability analysis of traveling waves

Recall that the TW solution of the KdV equation $u(x,t) = \phi(x - ct)$ is related to the Lax pair

$$\mathcal{L}(u)v = \lambda v, \quad \frac{\partial v}{\partial t} = \mathcal{M}(u)v,$$

for which we can separate the variables as $v(x,t) = w(x-ct)e^{\omega t}$.

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for which we can separate the variables as $v(x,t) = w(x - ct)e^{\omega t}$.

The “dispersion" relation $\omega = \omega(\lambda)$ can be found from the characteristic polynomial

$$\omega^2 + 16P(\lambda) = 0, \quad P(\lambda) = \lambda^3 + \frac{c}{2}\lambda^2 + \frac{c^2 - 4b}{16}\lambda - \frac{d + bc}{16},$$

which follows from the commutability of

$$\begin{aligned} w''(x) + \phi(x)w(x) + \lambda w(x) &= 0, \\ -3\phi'(x)w(x) - 6\phi w'(x) - 4w'''(x) + cw'(x) &= \omega w(x). \end{aligned}$$

State-of-art of stability analysis of traveling waves

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$$\mathcal{L}(u)v = \lambda v, \quad \frac{\partial v}{\partial t} = \mathcal{M}(u)v,$$

for which we can separate the variables as $v(x,t) = w(x-ct)e^{\omega t}$.

Linearized KdV equation at the TW with profile $\phi(x-ct)$ and perturbation $\mathfrak{u}(x-ct)e^{\Lambda t}$ can be solved by the squared eigenfunctions

$$\mathfrak{u}(x) = w(x)w'(x), \quad \Lambda = 2\omega = \pm 8i\sqrt{P(\lambda)},$$

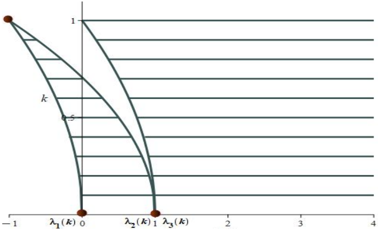
where $w(x)$ is the eigenfunction of the Lax system and

$$P(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3).$$

State-of-art of stability analysis of traveling waves

If $\phi(x+L) = \phi(x)$ is spatially periodic, then $w(x+L) = w(x)e^{i\kappa x}$ is required to be bounded which is only possible inside the spectral bands of the Lax spectrum:

$$\lambda \in \sigma_L := [\lambda_1, \lambda_2] \cup [\lambda_3, \infty).$$



Since $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) > 0$ for $\lambda \in \sigma_L$, then we have $\Lambda = \pm 8i\sqrt{P(\lambda)} \in i\mathbb{R}$, hence the TW is spectrally stable.

Section 2

Traveling waves and their spectral stability in the modified KdV equation

Traveling periodic waves in the modified KdV equation

The modified KDV equation is physically relevant for fluids. There are two meaningful cases

$$u_t \pm 6u^2 u_x + u_{xxx} = 0,$$

which are called focusing and defocusing, by the analogy to the NLS

$$i\psi_t + \psi_{xx} \pm 2|\psi|^2\psi = 0.$$

However, the NLS and mKdV equations are very different in the traveling wave solutions $u(x,t) = \phi(x - ct)$:

$$\phi''' \pm 6\phi^2\phi' - c\phi' = 0.$$

After integration(s), it yields

$$\phi'' \pm 2\phi^3 - c\phi = b \quad \Rightarrow \quad (\phi')^2 \pm \phi^4 - c\phi^2 = 2b\phi + d$$

Traveling periodic waves in the modified KdV equation

Since only one scaling transformation is available

$$\phi(x) = a\tilde{\phi}(ax), \quad c = a^2\tilde{c}, \quad b = a^3\tilde{b}, \quad d = a^4\tilde{d},$$

two parameters are independent.

If $b = 0$, the solutions are expressed in terms of Jacobi dn, cn, and sn functions like in NLS, e.g. in the defocusing case,

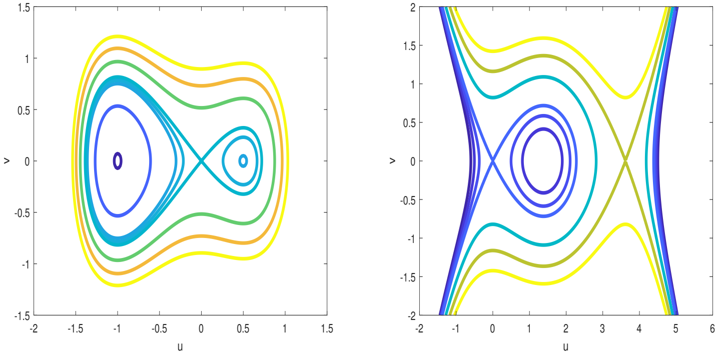
$$\phi(x) = k\,\mathrm{sn}(x,k), \quad c = 1 + k^2, \quad b = 0, \quad d = \frac{1}{2}k^2.$$

If $b \neq 0$, the solutions are expressed in elliptic functions,

$$\phi(x) = u_4 + \frac{(u_2 - u_4)(u_3 - u_4)}{(u_2 - u_4) - (u_2 - u_3)\mathrm{sn}^2(x,k)}, \quad k^2 = \frac{(u_1 - u_4)(u_2 - u_3)}{(u_1 - u_3)(u_2 - u_4)},$$

where $(\phi')^2 = (\phi - u_1)(\phi - u_2)(\phi - u_3)(\phi - u_4)$.

Traveling periodic waves in the modified KdV equation



Focusing case (left): two families exist to generalize cn and dn.
Defocusing case (right): one family exists to generalize sn.

Algebraic characterization of traveling waves

The mKdV equation is a compatibility condition of the Lax system

$$\partial_x \varphi = \begin{pmatrix} i\zeta & u \\ u & -i\zeta \end{pmatrix} \varphi$$

and

$$\partial_t \varphi = \left(\begin{array}{ll} 4i\zeta^3 + 2i\zeta u^2 & 4\zeta^2 u - 2i\zeta u_x + 2u^3 - u_{xx} \\ 4\zeta^2 u + 2i\zeta u_x + 2u^3 - u_{xx} & -4i\zeta^3 - 2i\zeta u^2 \end{array} \right) \varphi.$$

If $u(x,t) = \phi(x-ct)$, then $\varphi(x,t) = \psi(x-ct)e^{\omega t}$, where ω is found from the algebraic system

$$\begin{aligned} &\omega\psi - c \begin{pmatrix} i\zeta & \phi \\ \phi & -i\zeta \end{pmatrix} \psi \\ &= \left(\begin{array}{ll} 4i\zeta^3 + 2i\zeta\phi^2 & 4\zeta^2\phi - 2i\zeta\phi' - c\phi - b \\ 4\zeta^2\phi + 2i\zeta\phi' - c\phi - b & -4i\zeta^3 - 2i\zeta\phi^2 \end{array} \right) \psi. \end{aligned}$$

Algebraic characterization of traveling waves

The characteristic equation is

$$\omega^2 + 16P(\zeta) = 0, \quad P(\zeta) := \zeta^6 - \frac{c}{2}\zeta^2 + \frac{1}{16}(c^2 - 8d)\zeta^2 - \frac{b^2}{16},$$

which can be factorized as $P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$.

Lax spectrum σ_L for traveling periodic waves is the Floquet spectrum

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

where $\phi(x+L) = \phi(x)$ and $\psi(x+L) = e^{i\theta x}\psi(x)$ with $\theta \in \left[-\frac{\pi}{L}, \frac{\pi}{L}\right]$.

Algebraic characterization of traveling waves

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which can be factorized as $P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$.

Roots of $P(\zeta)$ correspond to either $\theta = 0$ or $\theta = \frac{\pi}{L}$ for which

$$\phi = p_1^2 + q_1^2 + p_2^2 + q_2^2,$$

where $\psi = (p_1, q_1)^T$ and $\psi = (p_2, q_2)^T$ are eigenvectors for two different roots of $P(\zeta)$.

C. Cao & X. Geng, (1990)

Jinbing Chen & D. P., J. Nonlinear Science (2019)

Algebraic characterization of traveling waves

The characteristic equation is

$$\omega^2 + 16P(\zeta) = 0, \quad P(\zeta) := \zeta^6 - \frac{c}{2}\zeta^2 + \frac{1}{16}(c^2 - 8d)\zeta^2 - \frac{b^2}{16},$$

which can be factorized as $P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$.

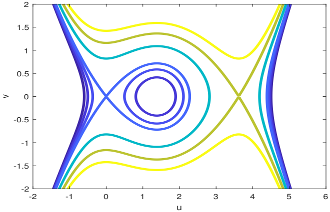
Linearized modified KdV equation at the traveling wave with profile $\phi(x - ct)$ under the perturbation $v(x - ct)e^{\Lambda t}$ can be solved in terms of squared eigenfunctions:

$$v = p^2 - q^2, \qquad \Lambda = 2\omega = \pm 8i\sqrt{P(\zeta)},$$

where $\psi = (p, q)^T$ is an eigenvector for $\zeta \in \sigma_L$.

B. Deconinck & M. Nivala, (2011) B. Deconinck & J. Upsal (2021)

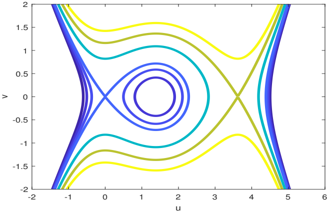
Spectral stability of traveling waves: defocusing case



$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Spectral stability of traveling waves: defocusing case



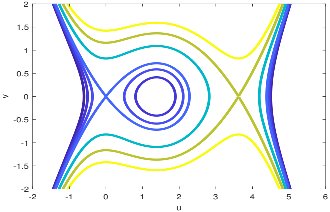
$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Since the spectral problem is self-adjoint, the Lax spectrum $\sigma_L \subset \mathbb{R}$.

$P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$ has real $0 \leq \zeta_3 \leq \zeta_2 \leq \zeta_1$.

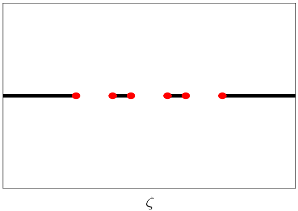
Spectral stability of traveling waves: defocusing case



$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Lax spectrum σ_L is

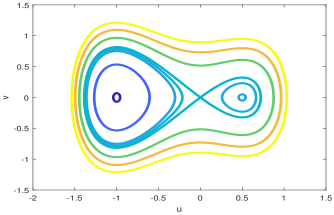


We have

$$P(\zeta) > 0 \qquad \zeta \in \sigma_L$$

hence
$\Lambda = \pm 8i\sqrt{P(\zeta)} \in i\mathbb{R}$ and
the periodic wave is
spectrally stable.

Spectral stability of traveling waves: focusing case

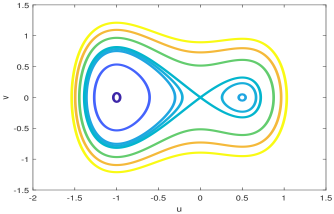


The spectral problem is

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Spectral stability of traveling waves: focusing case



The spectral problem is

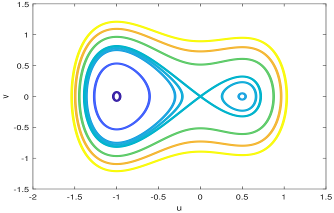
$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

The spectral problem is no longer self-adjoint and $\sigma_L \subset \mathbb{C}$.

$P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$ has complex $\zeta_{1,2,3} \in \mathbb{C}$.

Spectral stability of traveling waves: focusing case

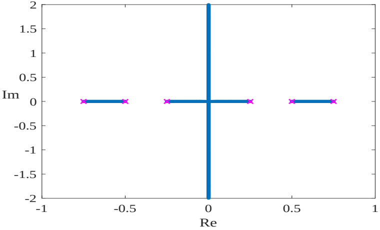


The spectral problem is

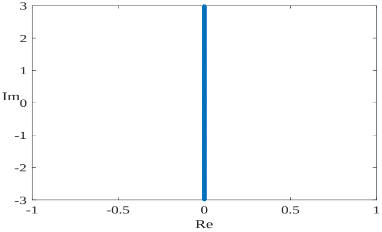
$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

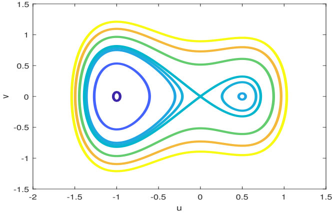
Lax spectrum σ_L for $\lambda = i\zeta$ is



Stability spectrum $\Lambda = \pm 8i\sqrt{P(\zeta)}$:



Spectral stability of traveling waves: focusing case



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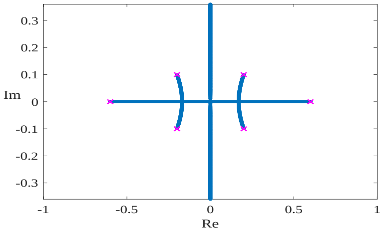

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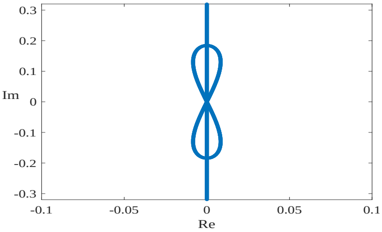

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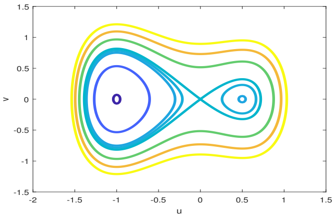

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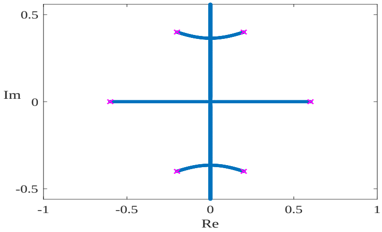

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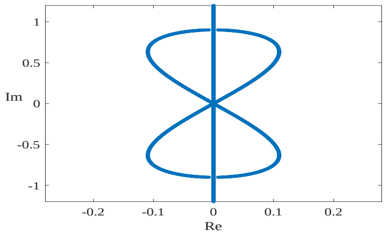


Section 3

Breathers on the traveling waves in the defocusing modified KdV equation

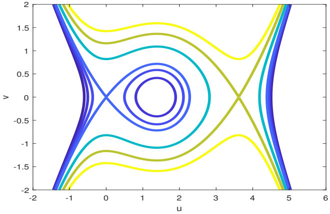
Recap of what was said before

Here we consider the defocusing MKDV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

with one family of traveling waves $u(x,t) = \phi(x+ct)$, $c > 0$ such that

$$(\phi')^2 = (\phi - u_1)(\phi - u_2)(\phi - u_3)(\phi - u_4) = \phi^4 - c\phi^2 + 2b\phi + d.$$



The spectral problem is:

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Recap of what was said before

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$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

with one family of traveling waves $u(x,t) = \phi(x+ct)$, $c > 0$ such that

$$(\phi')^2 = (\phi - u_1)(\phi - u_2)(\phi - u_3)(\phi - u_4) = \phi^4 - c\phi^2 + 2b\phi + d.$$

The general periodic solution is expressed in elliptic functions,

$$\phi(x) = u_4 + \frac{(u_2 - u_4)(u_3 - u_4)}{(u_2 - u_4) - (u_2 - u_3)\mathrm{sn}^2(x,k)}, \quad k^2 = \frac{(u_1 - u_4)(u_2 - u_3)}{(u_1 - u_3)(u_2 - u_4)},$$

with a nontrivial dependence between (u_1, u_2, u_3, u_4) and (b, c, d).

Recap of what was said before

Here we consider the defocusing MKDV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

with one family of traveling waves $u(x,t) = \phi(x+ct)$, $c > 0$ such that

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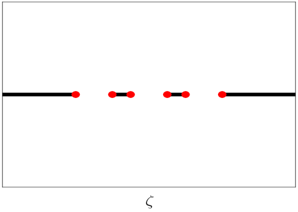
Lax spectrum σ_L is



We have

$$P(\zeta) > 0 \qquad \zeta \in \sigma_L$$

hence
$\Lambda = \pm 8i\sqrt{P(\zeta)} \in i\mathbb{R}$ and
the periodic wave is spectrally stable.

Recap of what was said before

Here we consider the defocusing MKDV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

with one family of traveling waves $u(x,t) = \phi(x+ct)$, $c > 0$ such that

$$(\phi')^2 = (\phi - u_1)(\phi - u_2)(\phi - u_3)(\phi - u_4) = \phi^4 - c\phi^2 + 2b\phi + d.$$

If $b = 0$, the solution is the same as in the NLS equation:

$$u(x,t) = \phi(x+ct), \quad \phi(x) = k\,\mathrm{sn}(x;k), \quad c = 1 + k^2, \quad k \in (0,1),$$

and the breathers have been constructed in the explicit form:

A. Mucalica & D. P., Lett. Math. Phys. (2024)

If $b \neq 0$, breathers have been constructed recently:

L.K. Arruda & D. P., J. Nonlinear Waves (2025)

D. P. & R. Weikard, Stud. Appl. Math. (2026)

Construction of dark breathers for $b = 0$

There exists an exact solution of the Lax system for $\psi = (p, q)^T$:

$$p = e^{sx+\omega t} e^{-\frac{i\pi x}{4K}} \frac{H(x-z)}{\Theta(x)\Theta(z)}, \qquad q = e^{sx+\omega t} e^{-\frac{i\pi x}{4K}} \frac{\Theta(x-z)}{\Theta(x)H(z)},$$

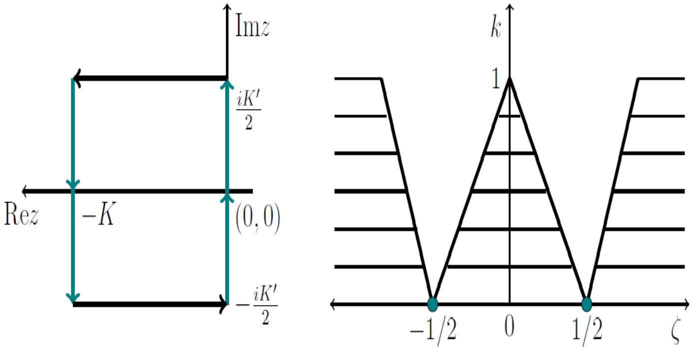
where $z \in \mathbb{C}$ is a smart choice of parameterization for which

$$\begin{aligned} \zeta &= \frac{1}{2}\,\mathrm{dn}(z)\,\mathrm{dn}(iK' - z), \\ s &= \frac{1}{2}Z(z) - \frac{1}{2}Z(iK' - z), \\ \omega^2 &= 16\zeta^2(\zeta_1^2 - \zeta^2)(\zeta^2 - \zeta_2^2), \end{aligned}$$

since $\zeta_3 = 0$ if $b = 0$.

[D. A. Takahashi, Phys. Rev. E (2016)]

Construction of dark breathers for $b = 0$



Lax spectrum $\sigma_L = (-\infty, -\zeta_1] \cup [-\zeta_2, \zeta_2] \cup [\zeta_1, \infty)$.

Construction of dark breathers for $b = 0$

Dark breathers correspond to a point ζ in the gap (ζ_2, ζ_1) for which we can parameterize

$$z = \frac{iK'}{2} - \alpha, \quad \alpha \in (0, K)$$

and compute

$$\zeta = \frac{1+k}{2} \, \frac{1 - k\,\mathrm{sn}^2(\alpha)}{1 + k\,\mathrm{sn}^2(\alpha)},$$

$$s = -Z(\alpha) - \frac{k\,\mathrm{sn}(\alpha)\,\mathrm{cn}(\alpha)\,\mathrm{dn}(\alpha)}{1 + k\,\mathrm{sn}^2(\alpha)} < 0,$$

$$\omega = -2k(1+k)^2 \frac{1 - k\,\mathrm{sn}^2(\alpha)}{[1 + k\,\mathrm{sn}^2(\alpha)]^3}\,\mathrm{sn}(\alpha)\,\mathrm{cn}(\alpha)\,\mathrm{dn}(\alpha) < 0.$$

Construction of dark breathers for $b = 0$

New solution is obtained with the Darboux transformation:

$$\hat{u} = u - \frac{4i\zeta pq}{p^2 - q^2},$$

where

$$p = \bar{q} = e^{-\eta} e^{-\frac{i\pi\xi}{4K}} \frac{H\left(\xi + \alpha - \frac{iK'}{2}\right)}{\Theta(\xi)\Theta\left(-\alpha + \frac{iK'}{2}\right)} + e^{\eta} e^{-\frac{i\pi\xi}{4K}} \frac{H\left(\xi - \alpha - \frac{iK'}{2}\right)}{\Theta(\xi)\Theta\left(\alpha + \frac{iK'}{2}\right)},$$

with $\xi := x + ct$ (periodic wave coordinate) and $\eta := -s\xi - \omega t + \eta_0$ (dark soliton coordinate). With (lots) of elliptic function relations (e.g. quarter-period translations of Jacobi theta's functions), one can obtain an exact solution for $\hat{u}(x,t)$.

However, the new solution $\hat{u}(x,t)$ is singular! A bounded solution is obtained after the transformation: $\tilde{u}(x,t) := \hat{u}(x + iK', t)$.

Construction of dark breathers for $b = 0$

If the traveling wave solution is

$$u(x,t) = k\,\mathrm{sn}(\xi;k) = \sqrt{k}\frac{H(\xi)}{\Theta(\xi)},$$

then the dark breather solution is

$$\tilde{u}(x,t) = \sqrt{k}\,\frac{H(\xi+2\alpha)e^{-2\eta} + H(\xi-2\alpha)e^{2\eta} + 2\beta H(\xi)}{\Theta(\xi+2\alpha)e^{-2\eta} + \Theta(\xi-2\alpha)e^{2\eta} + 2\gamma\Theta(\xi)},$$

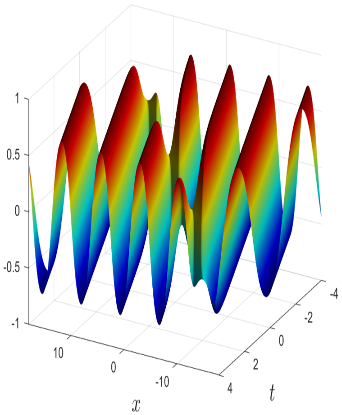
with some explicit $\beta, \gamma \in \mathbb{R}$ and $\xi = x + c_0 t$, $\eta = \kappa(x + ct + x_0)$.

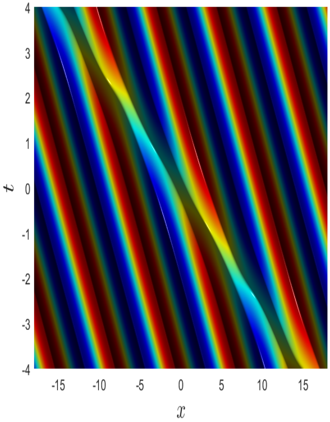
For $k = 1$, this yields the two-soliton solution:

$$\tilde{u} = \frac{\sinh(\xi+2\alpha)e^{-2\eta} + \sinh(\xi-2\alpha)e^{2\eta} + 2\sinh(\xi)(1-\sinh^2(2\alpha))\mathrm{sech}(2\alpha)}{\cosh(\xi+2\alpha)e^{-2\eta} + \cosh(\xi-2\alpha)e^{2\eta} + 2\cosh(\xi)\cosh(2\alpha)}.$$

A. Mucalica & D. P., Lett. Math. Phys. (2024)

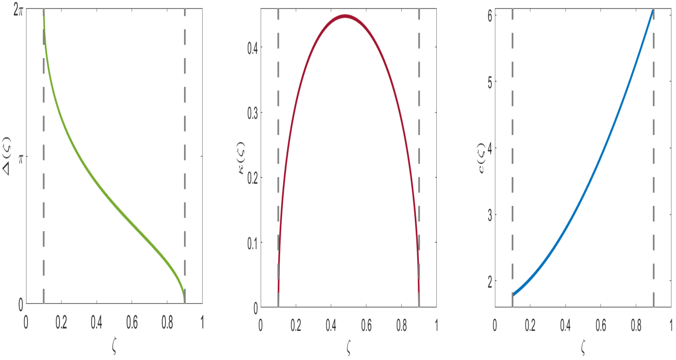
Construction of dark breathers for $b = 0$



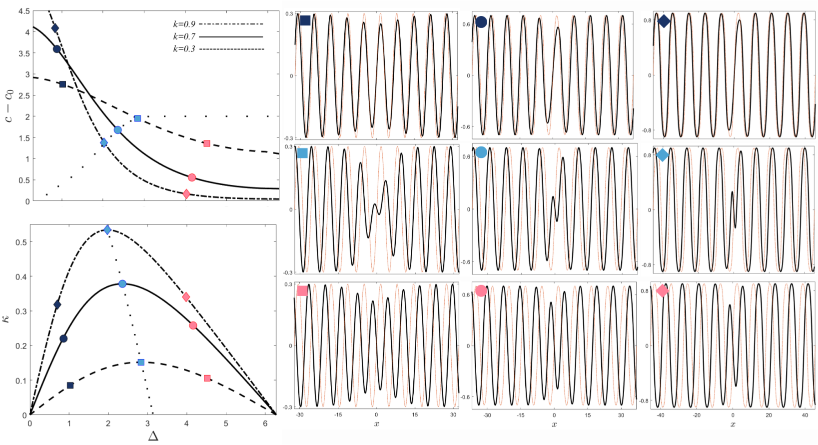


Construction of dark breathers for $b = 0$

Characteristics of the dark breather:

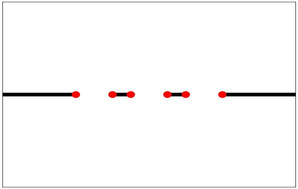


Construction of dark breathers for $b = 0$



Construction of breathers for $b \neq 0$

Lax spectrum σ_L is



ζ

$$P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_2^2)$$

with $0 \leq \zeta_3 < \zeta_2 < \zeta_1$.

It was shown by A. Kamchatnov (1999) that the periodic solution can be expressed by using $\zeta_1, \zeta_2, \zeta_3$:

$$\phi(x) = \frac{2(\zeta_1 + \zeta_3)(\zeta_2 + \zeta_3)}{(\zeta_1 + \zeta_3) - (\zeta_1 - \zeta_2)\mathrm{sn}^2(\nu x, k)} - \zeta_1 - \zeta_2 - \zeta_3, \quad k^2 = \frac{\zeta_1^2 - \zeta_2^2}{\zeta_1^2 - \zeta_3^2}.$$

where $\nu = \sqrt{\zeta_1^2 - \zeta_2^2}$. Transformation $(\zeta_1, \zeta_2, \zeta_3) \mapsto (b, c, d)$ is an invertible diffeomorphism for $0 \leq \zeta_3 < \zeta_2 < \zeta_1$.

Construction of breathers for $b \neq 0$

Weierstrass' elliptic function $\wp(x)$ is related to $\mathrm{sn}^2(x,k)$, hence $\phi(x)$ is a linear fractional transformation of $\wp(x)$. Moreover, it was known for at least 100 years [N.I. Akhiezer (1970), E.T. Whittaker–G.N. Watson (1927)] that

$$\phi^2(x) = \wp(x+v) + \wp(x-v) + \wp(2v).$$

and

$$\phi'(x) = \wp(x-v) - \wp(x+v),$$

where $\pm v$ are double poles of the elliptic functions $\phi^2(x)$ and $\phi'(x)$ inside the fundamental rectangle

$$\mathcal{R} := \left\{ z \in \mathbb{C} : \quad -K \leq \mathrm{Re}(z) \leq K, \;\; -K' \leq \mathrm{Im}(z) \leq K' \right\},$$

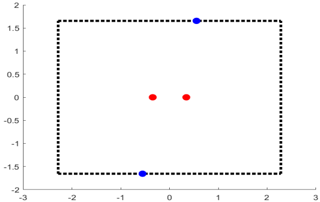
where $z = \nu x$.

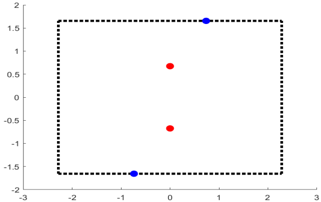
Construction of breathers for $b \neq 0$

This brings up theory of elliptic functions:

- Numbers of zeros and poles of $\phi(x)$ are equal in $\mathcal{R}$.
- The number of zeros and poles of $\wp(x)$ is equal to 2 in $\mathcal{R}$.
- Every elliptic function can be factorized as a quotient of the product of Jacobi's theta function $H(x)$.

We proved that poles of $\phi(x)$ are $\pm(iK' + \alpha) = \pm\nu v$ with $\alpha \in (0, K)$ and the zero of $\phi(x)$ are $\pm\beta$ with either $\beta \in (0, K)$ or $\beta \in (0, iK')$.





Construction of breathers for $b \neq 0$

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- The number of zeros and poles of $\wp(x)$ is equal to 2 in $\mathcal{R}$.
- Every elliptic function can be factorized as a quotient of the product of Jacobi's theta function $H(x)$.

This yields the new representation of periodic solutions:

$$\phi(x) = (\zeta_1 - \zeta_2 - \zeta_3) \frac{H(\nu x - \beta) H(\nu x + \beta) \Theta^2(\alpha)}{\Theta(\nu x - \alpha) \Theta(\nu x + \alpha) H^2(\beta)},$$

where $\alpha \in (0, K)$ and either $\beta \in (0, K)$ or $\beta \in (0, iK')$.

L.K. Arruda & D. P., J. Nonlinear Waves (2025)

Construction of breathers for $b \neq 0$

This brings up theory of elliptic functions:

- Numbers of zeros and poles of $\phi(x)$ are equal in $\mathcal{R}$.
- The number of zeros and poles of $\wp(x)$ is equal to 2 in $\mathcal{R}$.
- Every elliptic function can be factorized as a quotient of the product of Jacobi's theta function $H(x)$.

The symmetric case $b = 0$ (for which $\zeta_3 = 0$) is different since it is not a particular limit of the general periodic solution and the elliptic modulus k is different too. The snoidal solution $\phi(x) = \tilde{k}\,\mathrm{sn}(x, \tilde{k})$ is obtained from the general periodic solution with Landen transformation:

$$k = \frac{2\sqrt{\tilde{k}}}{1 + \tilde{k}}.$$

Eigenfunctions of the Lax system for $b \neq 0$

We take the elliptic traveling wave solution

$$\phi(x) = (\zeta_1 - \zeta_2 - \zeta_3) \frac{H(\nu x - \beta) H(\nu x + \beta) \Theta^2(\alpha)}{\Theta(\nu x - \alpha) \Theta(\nu x + \alpha) H^2(\beta)}$$

and attempt to construct elliptic solutions of the Lax system:

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x).$$

Eigenfunctions of the Lax system for $b \neq 0$

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$$\phi(x) = (\zeta_1 - \zeta_2 - \zeta_3) \frac{H(\nu x - \beta) H(\nu x + \beta) \Theta^2(\alpha)}{\Theta(\nu x - \alpha) \Theta(\nu x + \alpha) H^2(\beta)}$$

and attempt to construct elliptic solutions of the Lax system:

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x).$$

For $\zeta = 0$, we found two explicit solutions:

$$\psi_1 = e^{-\frac{H'(2\alpha)}{H(2\alpha)} \nu x} \frac{\Theta(\nu x + \alpha)}{\Theta(\nu x - \alpha)} \left(\begin{array}{c} 1 \\ -1 \end{array} \right), \quad \psi_2 = e^{\frac{H'(2\alpha)}{H(2\alpha)} \nu x} \frac{\Theta(x - \alpha)}{\Theta(x + \alpha)} \left(\begin{array}{c} 1 \\ 1 \end{array} \right),$$

with the time dependence given by $\omega = \pm 4 \zeta_1 \zeta_2 \zeta_3$.

Kink breathers for $b \neq 0$ and $\zeta = 0$

By using the Darboux transformation in the limit $\zeta \to 0$:

$$\hat{u} = u - \frac{4i\zeta pq}{p^2 - q^2},$$

we obtained the new solution:

$$\hat{u}(x,t) = \frac{4\phi(\xi)\Theta^2(\nu\xi + \alpha) + e^{2(\eta+\eta_0)}\Theta^2(\nu\xi - \alpha)(2\phi(\xi)\phi'(\xi) - \phi''(\xi) - b)}{4\Theta^2(\nu\xi + \alpha) + e^{2(\eta+\eta_0)}\Theta^2(\nu\xi - \alpha)(c + 2\phi'(\xi) - 2\phi(\xi)^2)},$$

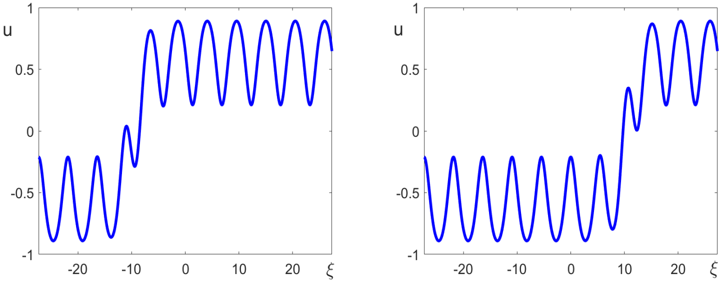
where $\xi = x + ct$, $\eta = \frac{H'(2\alpha)}{H(2\alpha)}(x + c_b t)$, and $c_b < c$, such that

$$\hat{u}(x,t) \to \begin{cases} \phi(\xi) & \text{as } \eta \to -\infty, \\ -\phi(\xi - 2\nu^{-1}\alpha) & \text{as } \eta \to +\infty, \end{cases}$$

L.K. Arruda & D. P., J. Nonlinear Waves (2025)

Kink breathers for $b \neq 0$ and $\zeta = 0$

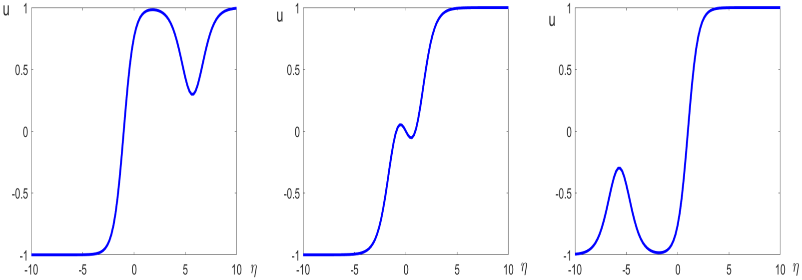
This new solution defines the kink breather:



Kink breathers for $b \neq 0$ and $\zeta = 0$

In the limit $k \to 1$ (that is for $\zeta_2 = \zeta_3$) the kink breather recovers the two-soliton solution, which is opposite of the dark breather:

$$\hat{u} = \frac{\sinh(\eta + 2\alpha)e^{-2\nu\xi} + \sinh(\eta - 2\alpha)e^{2\nu\xi} + 2\sinh(\eta)(1 - \sinh^2(2\alpha))\mathrm{sech}(2\alpha)}{\cosh(\eta + 2\alpha)e^{-2\nu\xi} + \cosh(\eta - 2\alpha)e^{2\nu\xi} + 2\cosh(\eta)\cosh(2\alpha)}.$$



Eigenfunctions of the Lax system for $b \neq 0$ and $\zeta \neq 0$

We take the elliptic traveling wave solution

$$\phi(x) = (\zeta_1 - \zeta_2 - \zeta_3) \frac{H(\nu x - \beta) H(\nu x + \beta) \Theta^2(\alpha)}{\Theta(\nu x - \alpha) \Theta(\nu x + \alpha) H^2(\beta)}$$

and attempt to construct elliptic solutions of the Lax system:

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x).$$

Eigenfunctions of the Lax system for $b \neq 0$ and $\zeta \neq 0$

For $\zeta \neq 0$, by using [Belokolos–Bobenko-Enolskii–Its–Matveev, 1994] we can write for $\psi = (p, q)^T$:

$$
\begin{aligned}
\frac{q}{p} &= -i\frac{\zeta^3 + \frac{1}{2}\zeta(\phi^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2) - \sqrt{P(\zeta)}}{\zeta^2\phi - \frac{i}{2}\zeta\phi' - \zeta_1\zeta_2\zeta_3} \\
&= -i\frac{\zeta^2\phi + \frac{i}{2}\zeta\phi' - \zeta_1\zeta_2\zeta_3}{\zeta^3 + \frac{1}{2}\zeta(\phi^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2) + \sqrt{P(\zeta)}} \\
&= C\frac{H(x + z_1^*)H(x + z_2^*)}{H(x - z_1)H(x - z_2)},
\end{aligned}
$$

where $\{\pm z_1, \pm z_2\}$ is the set of zeros of the second denominator and $\{\pm z_1^*, \pm z_2^*\}$ is the set of zeros of the first numerator such that

$$
z_1 + z_2 + z_1^* + z_2^* = 0 \bmod(2K, 2K').
$$

Eigenfunctions of the Lax system for $b \neq 0$ and $\zeta \neq 0$

With further integration of

$$p'(x) = i\zeta p(x) + \phi(x)q(x),$$

we can obtain

$$p = e^{sx+\omega t}\frac{H(x-z_1)H(x-z_2)}{\Theta(x-\alpha)\Theta(x+\alpha)\Theta(\alpha-z_1)\Theta(\alpha-z_2)}e^{-\frac{i\pi}{2K}(z_1+z_2)},$$

$$q = -e^{sx+\omega t}\frac{H(x+z_1^*)H(x+z_2^*)}{\Theta(x-\alpha)\Theta(x+\alpha)\Theta(\alpha+z_1^*)\Theta(\alpha+z_2^*)}e^{\frac{i\pi}{2K}(z_1^*+z_2^*)},$$

with unique expression for s and $\omega = 4i\sqrt{P(\zeta)}$.

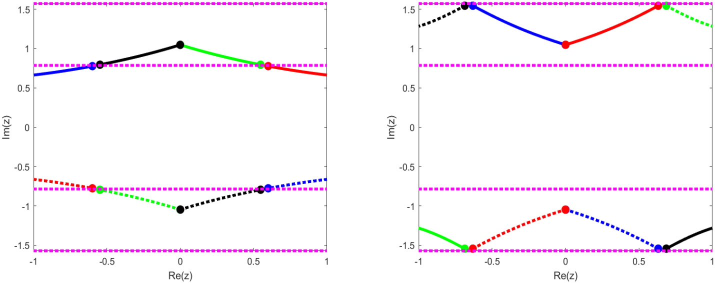
However, this method does not tell us how to parameterize $\{z_1, z_2, z_1^*, z_2^*\}$ in terms of $\zeta \in \mathbb{R}$.

Eigenfunctions of the Lax system for $b \neq 0$ and $\zeta \neq 0$

Numerical results in the hyperbolic case $\zeta_2 = \zeta_3$ suggested that

$$z_1 = -\bar{z}_1^*, \qquad z_2 = -\bar{z}_2^*,$$

which still give three real parameters defined uniquely from $\zeta \in \mathbb{R}$.



Roots $\{\pm z_1, \pm z_2\}$ (blue,green) and $\{\pm z_1^*, \pm z_2^*\}$ (red,black) parameterized by ζ in (ζ_2, ζ_1) (left) and $(0, \zeta_3)$ (right).

Finding eigenfunctions for $b \neq 0$ and $\zeta \neq 0$

Recall the relation to Weierstrass's elliptic functions:

$$\phi^2(x) = \wp(x+v) + \wp(x-v) + \wp(2v).$$

and

$$\phi'(x) = \wp(x-v) - \wp(x+v),$$

where $\pm v = \pm(iK' + \alpha)/\nu$ are double poles of $\phi^2(x)$ and $\phi'(x)$.

Finding eigenfunctions for $b \neq 0$ and $\zeta \neq 0$

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Squaring the Lax system yields

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x) \Rightarrow \begin{pmatrix} -\partial_x^2 + \phi^2 & \phi' \\ \phi' & -\partial_x^2 + \phi^2 \end{pmatrix} \psi = \zeta^2 \psi.$$

If $\psi = (p,q)^T$ and $\psi_\pm = p \pm q$, this leads to

$$\left(-\partial_x^2 + \phi^2 \pm \phi'\right)\psi_\pm = \zeta^2 \psi_\pm, \quad \phi^2(x) \pm \phi'(x) = \wp(v) + 2\wp\left(x \mp \frac{v}{2}\right).$$

Finding eigenfunctions for $b \neq 0$ and $\zeta \neq 0$

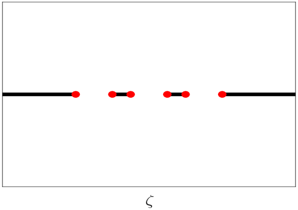
This folding represents the Miura transformation, which connects elliptic solutions of the mKdV and KdV equations.

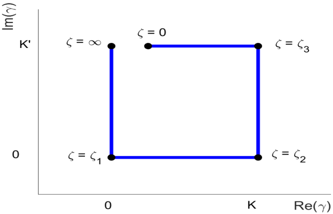
This yields the explicit eigenfunctions:

$$\psi_\pm(x) = c_1^\pm \frac{H(\nu x + \gamma \pm \alpha)}{\Theta(\nu x \pm \alpha)} e^{-\nu x Z(\gamma)} + c_2^\pm \frac{H(\nu x - \gamma \pm \alpha)}{\Theta(\nu x \pm \alpha)} e^{\nu x Z(\gamma)},$$

with parameterization of $\zeta \in \mathbb{R}$ by $\gamma \in \mathbb{C}$:

$$\zeta^2 = \zeta_1^2 - (\zeta_1^2 - \zeta_2^2)\,\mathrm{sn}^2(\gamma) = \zeta_2^2 + (\zeta_1^2 - \zeta_2^2)\,\mathrm{cn}^2(\gamma) = \zeta_3^2 + (\zeta_1^2 - \zeta_3^2)\,\mathrm{dn}^2(\gamma).$$





Finding eigenfunctions for $b \neq 0$ and $\zeta \neq 0$

The coefficients $c_1^{\pm}$ and $c_2^{\pm}$ are found from the Lax system at the poles $\phi(x) \sim \mp \frac{1}{x - \frac{\nu}{2}}$ as $x \to \pm \frac{\nu}{2}$:

$$c_1^+ = -i\zeta, \quad c_1^- = \zeta_1 \frac{\Theta(0)\Theta(2\alpha + \gamma)}{\Theta(\gamma)\Theta(2\alpha)},$$
$$c_2^+ = \zeta_1 \frac{\Theta(0)\Theta(2\alpha + \gamma)}{\Theta(\gamma)\Theta(2\alpha)}, \quad c_2^- = i\zeta.$$

The time evolution in $\psi(x,t) = \psi_{1,2}(x + ct)e^{\pm\omega t}$ is obtained at the point $x_0 = \pm(\alpha - \gamma)/\nu$ for which $p(x_0) = \pm q(x_0)$. This yields

$$\omega = -4\nu^3 k^2 \,\mathrm{sn}(\gamma)\,\mathrm{cn}(\gamma)\,\mathrm{dn}(\gamma).$$

Finding eigenfunctions for $b \neq 0$ and $\zeta \neq 0$

Thus, we obtain the fully explicit expressions for eigenfunctions:

$$\psi(x,t) = \begin{pmatrix} p_1(x+ct) \\ q_1(x+ct) \end{pmatrix} e^{\omega t}, \quad \psi(x,t) = \begin{pmatrix} p_2(x+ct) \\ q_2(x+ct) \end{pmatrix} e^{-\omega t},$$

with

$$\begin{pmatrix} p_1(x) \\ q_1(x) \end{pmatrix} = \begin{pmatrix} -i\zeta \frac{H(\nu x+\gamma+\alpha)}{\Theta(\nu x+\alpha)} + \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{H(\nu x+\gamma-\alpha)}{\Theta(\nu x-\alpha)} \\ -i\zeta \frac{H(\nu x+\gamma+\alpha)}{\Theta(\nu x+\alpha)} - \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{H(\nu x+\gamma-\alpha)}{\Theta(\nu x-\alpha)} \end{pmatrix} e^{-\nu x Z(\gamma)}$$

and

$$\begin{pmatrix} p_2(x) \\ q_2(x) \end{pmatrix} = \begin{pmatrix} \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{H(\nu x-\gamma+\alpha)}{\Theta(\nu x+\alpha)} + i\zeta \frac{H(\nu x-\gamma-\alpha)}{\Theta(\nu x-\alpha)} \\ \zeta_1 \frac{\Theta(0)\Theta(2\alpha+\gamma)}{\Theta(\gamma)\Theta(2\alpha)} \frac{H(\nu x-\gamma+\alpha)}{\Theta(\nu x+\alpha)} - i\zeta \frac{H(\nu x-\gamma-\alpha)}{\Theta(\nu x-\alpha)} \end{pmatrix} e^{\nu x Z(\gamma)},$$

for $\zeta^2 = \zeta_1^2 - (\zeta_1^2 - \zeta_2^2)\,\mathrm{sn}^2(\gamma)$ and $\omega = -4\nu^3 k^2\,\mathrm{sn}(\gamma)\,\mathrm{cn}(\gamma)\,\mathrm{dn}(\gamma)$.

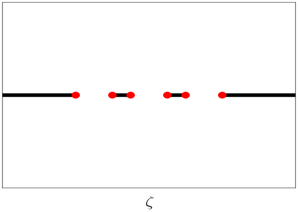
Construction of breathers for $b \neq 0$

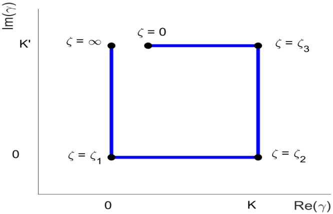
New solution is obtained with the Darboux transformation:

$$\hat{u} = u - \frac{4i\zeta pq}{p^2 - q^2},$$

where $p = c_1 p_1 + c_2 p_2$ and $q = c_1 q_1 + c_2 q_2$. which yields

$$u = \phi - \frac{4i\zeta(c_1^2 p_1 q_1 + c_1 c_2 (p_1 q_2 + p_2 q_1) + c_2^2 p_2 q_2)}{c_1^2 (p_1^2 - q_1^2) + 2c_1 c_2 (p_1 p_2 - q_1 q_2) + c_2^2 (p_2^2 - q_2^2)}.$$



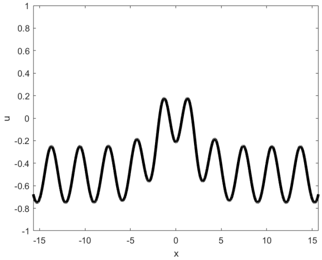


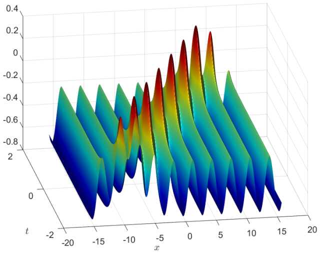
Construction of breathers for $b \neq 0$

If $\zeta \in (0, \zeta_3)$, then $\gamma = iK' + \delta$ with $\delta \in (2\alpha, K)$ and the solution is real-valued and bounded for

$$c_1 = e^{-\nu\kappa x_0}, \quad c_2 = -ie^{\nu\kappa x_0}, \quad \kappa = \frac{H'(\delta)}{H(\delta)}.$$

This is a bright breather for $(\zeta_1, \zeta_2, \zeta_3) = (1.25, 1, 0.5)$.
It transforms to a kink breather as $\zeta \to 0$.





D. P. & R. Weikard, Stud. Appl. Math. (2026)

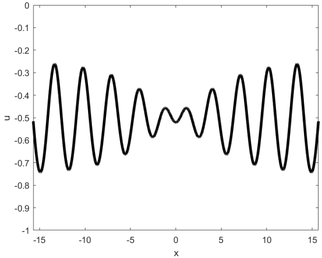
Construction of breathers for $b \neq 0$

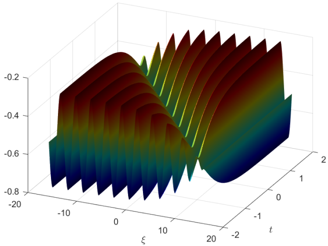
If $\zeta \in (\zeta_2, \zeta_1)$, then $\gamma \in (0, K)$ and the solution is real-valued for

$$c_1 = e^{-\nu\kappa x_0}, \quad c_2 = -ie^{\nu\kappa x_0}, \quad \kappa = \frac{\Theta'(\gamma)}{\Theta(\gamma)}.$$

It is singular for $x \in \mathbb{R}$ but becomes bounded after $x \to x + iK'$.

This is a dark breather for $(\zeta_1, \zeta_2, \zeta_3) = (1.25, 1, 0.5)$.





D. P. & R. Weikard, Stud. Appl. Math. (2026)

Conclusion

The following questions were discussed for elliptic traveling waves:

- How to connect spectral stability and Lax spectrum.
- How to characterize eigenfunctions of Lax system.
- How to construct breathers on the traveling wave background.

The new construction of elliptic eigenfunctions is expected to be useful in the focusing case both for the stable (dnoidal) and unstable (cnoidal) traveling wave background:

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Many thanks for your attention! Questions?