Thomas–Fermi ground state in a parabolic trap

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Introduction

Density waves in cigar–shaped Bose–Einstein condensates are modeled by the Gross-Pitaevskii equation

$$iu_t + \varepsilon^2 u_{xx} + (1 - x^2)u - |u|^2 u = 0,$$

where ε is a small parameter.

Limit $\varepsilon \to 0$ is referred to as the semi-classical limit or the Thomas–Fermi approximation since the work of L.H. Thomas (1927) and E. Fermi (1928).

Theorem(Ignat & Milot, 2006): For sufficiently small $\varepsilon > 0$, there exists a real-valued, positive-definite global minimizer of the Gross–Pitaevskii energy

$$E_\varepsilon(u) = \int_{\mathbb{R}} \left(\frac{1}{2} \varepsilon^2 |u_x|^2 + \frac{1}{2}(x^2 - 1)|u|^2 + \frac{1}{4}|u|^4 \right) dx$$

in the energy space

$$\mathcal{H}_1 = \left\{ u \in H^1(\mathbb{R}) : \ \ xu \in L^2(\mathbb{R}) \right\}.$$

Ground state in the variational theory

Let η_ε be a global minimizer of E_ε. From Euler–Lagrange equations, it solves

$$-\varepsilon^2 \eta_\varepsilon''(x) + \left(\eta_\varepsilon^2 + x^2 - 1\right) \eta_\varepsilon(x) = 0, \quad \forall x \in \mathbb{R}.$$

The formal limit for the ground state is

$$\eta_0(x) = \begin{cases} (1 - x^2)^{1/2}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1, \end{cases}$$

By variational analysis via sub- and super-solutions, it is true that

$$\begin{cases} 0 \leq \eta_\varepsilon(x) \leq C\,\varepsilon^{1/3} \exp\left(\frac{1-x^2}{4\,\varepsilon^{2/3}}\right) & \text{for } |x| \geq 1, \\ 0 \leq (1-x^2)^{1/2} - \eta_\varepsilon(x) \leq C\,\varepsilon^{1/3}(1-x^2)^{1/2} & \text{for } |x| \leq 1 - \varepsilon^{1/3}, \end{cases}$$

where C is ε-independent.

Ground state in the asymptotic theory

- Asymptotic solution is constructed on the three scales:

$$|x| \leq 1 - \varepsilon^{2/3}, \quad |x| \in (1 - \varepsilon^{2/3}, 1 + \varepsilon^{2/3}), \quad \text{and} \quad |x| \geq 1 + \varepsilon^{2/3}.$$

  with the WKB solutions, Painleve solutions, and Airy function solutions.

- Let

$$\eta_\varepsilon(x) = \varepsilon^{1/3}\, \nu_\varepsilon(y), \quad y = \frac{1 - x^2}{\varepsilon^{2/3}}$$

  and write an equation on $\eta_\varepsilon(y)$:

$$4(1 - \varepsilon^{2/3} y)\nu_\varepsilon''(y) - 2\,\varepsilon^{2/3}\, \nu_\varepsilon'(y) + y\nu_\varepsilon(y) - \nu_\varepsilon^3(y) = 0, \quad y \in (-\infty, \varepsilon^{-2/3}).$$

- The formal limit $\varepsilon \to 0$ gives the Painleve–II equation

$$4\nu''(y) + y\nu(y) - \nu^3(y) = 0, \quad y \in \mathbb{R},$$

  that admits a unique Hasting–McLeod (1986) solution $\nu_0(y)$ satisfying

$$\nu_0(y) \sim y^{1/2} \quad \text{as} \quad y \to +\infty \quad \text{and} \quad \nu_0(y) \to 0 \quad \text{as} \quad y \to -\infty.$$

Boscolo, et al. (2002); Konotop & Kevrekidis (2003); Zezyulin et al. (2008)

Spectral stability

Linearization of the Gross–Pitaevskii equation with

$$u(x,t) = \eta_\varepsilon(x) + [u(x) + iw(x)]\, e^{\lambda t} + [\bar{u}(x) - i\bar{w}(x)]\, e^{\bar{\lambda} t} + \mathcal{O}(\|u\|^2 + \|w\|^2)$$

results in the non-self-adjoint eigenvalue problem

$$\left\{ \begin{array}{rcr} -\varepsilon^2 u'' + (x^2 - 1 + 3\eta_\varepsilon^2)u & = & -\lambda w, \\ -\varepsilon^2 w'' + (x^2 - 1 + \eta_\varepsilon^2)w & = & \lambda u, \end{array} \right.$$

or, equivalently, in the generalized eigenvalue problem

$$\left(-\varepsilon^2 \partial_x^2 + x^2 - 1 + \eta_\varepsilon^2\right) w = \gamma \left(-\varepsilon^2 \partial_x^2 + x^2 - 1 + 3\eta_\varepsilon^2\right)^{-1} w,$$

where $\gamma = -\lambda^2$.

Eigenvalues in the formal Thomas–Fermi limit

- Restrict the generalized eigenvalue problem on $(-1,1)$ and drop ε-dependent terms in the right hand side:

$$\left(-\varepsilon^2 \partial_x^2 + x^2 - 1 + \eta_\varepsilon^2\right) w = \frac{\gamma w}{2(1-x^2)}, \quad x \in (-1,1).$$

- Let $\gamma = 2\varepsilon^2 \Gamma$ and use the definition of η_ε in the left hand side:

$$-w''(x) + \frac{\eta_\varepsilon''(x) w(x)}{\eta_\varepsilon(x)} = \frac{\Gamma w(x)}{(1-x^2)}, \quad x \in (-1,1).$$

- Substitution of $w(x) = v(x)\eta_\varepsilon(x)$ and taking the limit $\varepsilon \to 0$ result in the Legendre equation

$$-(1-x^2)v''(x) + 2xv'(x) = \Gamma v(x), \quad x \in (-1,1),$$

with eigenvalues at $\Gamma = n(n+1)$, $n \in \mathbb{N}$.

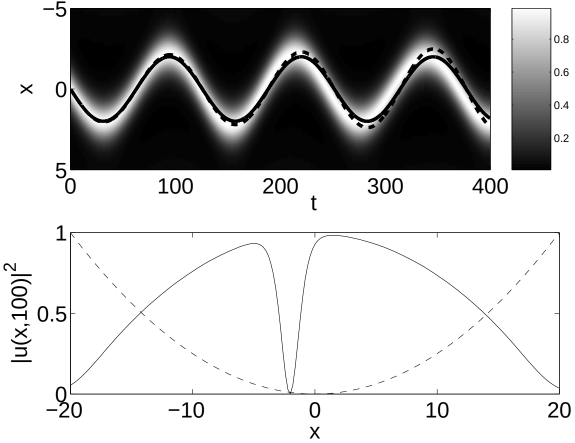
Stringari (1996); Fliesser et al. (1997); Eberlein et al. (2005)

Main objectives and results

- Obtain the uniform asymptotic approximation of the ground state η_ε in terms of solutions of the Painleve–II equation
- Study distribution of eigenvalues of the spectral stability for small $\varepsilon > 0$
- Extend the results to excited states with zeros on $\mathbb{R}$ that includes “one-dimensional vortices” (dark solitons).

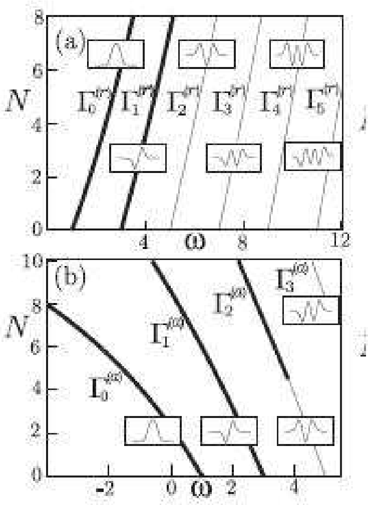
Gallo & P., J. Math. Anal. Appl. **355**, 495 (2009)
Gallo & P., preprint (2009).

Possible application: oscillations of 1-dim vortices



P. & Kevrekidis, Cont.Math. **473**, 159 (2008)
P. & Kevrekidis, ZAMP **59**, 559 (2008)

Possible application: stability of *m*-node vortices



Zezulin, Alfimov, Konotop, & Perez–Garcia, PRA (2008)

Asymptotic construction of the ground state

Let

$$\eta_\varepsilon(x) = \varepsilon^{1/3}\,\nu_\varepsilon(y), \quad y = \frac{1-x^2}{\varepsilon^{2/3}}$$

and write an equation on $\eta_\varepsilon(y)$:

$$4(1-\varepsilon^{2/3}\,y)\nu_\varepsilon''(y) - 2\,\varepsilon^{2/3}\,\nu_\varepsilon'(y) + y\nu_\varepsilon(y) - \nu_\varepsilon^3(y) = 0, \quad y \in J_\varepsilon,$$

where

$$J_\varepsilon := (-\infty, \varepsilon^{-2/3})$$

and $\nu_\varepsilon(y)$ decays to zero as $y \to -\infty$ and satisfies the Neumann boundary condition at $\varepsilon^{-2/3}$:

$$\eta_\varepsilon'(0) = 0 \quad \Longleftrightarrow \quad \lim_{y \uparrow \varepsilon^{-2/3}} \sqrt{1-\varepsilon^{2/3}\,y}\,\nu_\varepsilon'(y) = 0.$$

Asymptotic construction of the ground state

Fix $N \geq 0$ and look for solutions in the form

$$\nu_\varepsilon(y) = \sum_{n=0}^{N} \varepsilon^{2n/3}\, \nu_n(y) + \varepsilon^{2(N+1)/3}\, R_{N,\varepsilon}(y), \quad y \in J_\varepsilon,$$

where

- ν_0 solves the Painlevé-II equation

$$4\nu_0''(y) + y\nu_0(y) - \nu_0^3(y) = 0, \quad y \in \mathbb{R},$$

- for $1 \leq n \leq N$, ν_n solves

$$M_0\nu_n := -4\nu_n''(y) + \left(3\nu_0^2(y) - y\right)\nu_n(y) = F_n(y), \quad y \in \mathbb{R},$$

- $R_{N,\varepsilon}$ solves

$$-4(1-\varepsilon^{2/3}\, y)R_{N,\varepsilon}'' + 2\,\varepsilon^{2/3}\, R_{N,\varepsilon}' + \left(3\nu_0^2(y) - y\right) R_{N,\varepsilon} = F_{N,\varepsilon}(y, R_{N,\varepsilon}), \quad y \in J_\varepsilon,$$

Note: $\nu_n(y)$ does not depend on ε and is defined on $\mathbb{R}$, whereas the remainder term $R_{N,\varepsilon}$ is only defined on J_ε.

Main result

Theorem

Let ν_0 be the unique solution of the Painlevé II equation such that

$$\nu_0(y) \sim y^{1/2} \quad \text{as} \quad y \to +\infty \quad \text{and} \quad \nu_0(y) \to 0 \quad \text{as} \quad y \to -\infty.$$

For $n \geq 1$, ν_n is the unique solution of the linearized Painlevé equation in $\mathcal{C}^2(\mathbb{R}) \cap L^2(\mathbb{R})$. For every $N \geq 0$, there exists $\varepsilon_N > 0$ and $C_N > 0$ such that for every $0 < \varepsilon < \varepsilon_N$, there is

$$R_{N,\varepsilon} \in L^\infty(J_\varepsilon), \quad \text{with} \quad \|R_{N,\varepsilon}\|_{L^\infty(J_\varepsilon)} \leq C_N, \quad \lim_{y \to -\infty} R_{N,\varepsilon}(y) = 0,$$

such that for every $x \in \mathbb{R}$,

$$\eta_\varepsilon(x) = \varepsilon^{1/3} \sum_{n=0}^{N} \varepsilon^{2n/3} \nu_n \left(\frac{1 - x^2}{\varepsilon^{2/3}} \right) + \varepsilon^{2N/3+1} R_{N,\varepsilon} \left(\frac{1 - x^2}{\varepsilon^{2/3}} \right).$$

Step I: Hasting-McLeod solution

Ref: Fokas, Its, Kapaev, Novokshenov, AMS Monographs (2006)

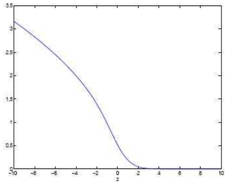
The Painlevé-II equation

$$4\nu''(y) + y\nu(y) - \nu^3(y) = 0, \quad y \in \mathbb{R},$$

admits a unique solution $\nu_0 \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\nu_0(y) = \frac{1}{2\sqrt{\pi}}(-2y)^{-1/4} e^{-\frac{2}{3}(-2y)^{3/2}} \left(1 + \mathcal{O}(|y|^{-3/4})\right) \underset{y\to-\infty}{\approx} 0,$$

$$\nu_0(y) \underset{y\to+\infty}{\approx} y^{1/2} \sum_{n=0}^{\infty} \frac{b_n}{(2y)^{3n/2}}.$$



12. Hastings-McLeod solution of the Painlevé II equation.

Step II: Linearized Painlevé-II equation

Let us consider the operator M_0 on $L^2(\mathbb{R})$, defined by

$$M_0 := -4\partial_y^2 + W_0(y), \quad W_0(y) = 3\nu_0^2(y) - y.$$

From the asymptotic behaviors of $\nu_0(y)$ as $y \to \pm\infty$, we infer that

$$W_0(y) \sim 2y \quad \text{as} \quad y \to +\infty \quad \text{and} \quad W_0(y) \sim -y \quad \text{as} \quad y \to -\infty.$$

Moreover, we prove that

$$\inf_{y \in \mathbb{R}} W_0(y) > 0$$

and $W_0(y)$ has the only extremum at the global minimum near $y = 0$.

For any $n \in \{1, 2, ..., N\}$, corrections $\nu_n \in \mathcal{C}^2(\mathbb{R}) \cap L^2(\mathbb{R})$ are found from the inhomogeneous equations $M_0 \nu_n = f_n$ such that

$$\nu_n(y) \underset{y \to +\infty}{\approx} y^{-5/2-2n} \sum_{m=0}^{\infty} g_{n,m} y^{-3m/2}, \quad \nu_n(y) \underset{y \to -\infty}{\approx} 0.$$

Step III: Remainder term

The remainder term satisfies

$$T^\varepsilon R_{N,\varepsilon}(y) = \frac{F_{N,\varepsilon}(y, R_{N,\varepsilon})}{\sqrt{1-\varepsilon^{2/3}y}}, \quad y \in J_\varepsilon,$$

where

$$T^\varepsilon = -4\partial_y \sqrt{1-\varepsilon^{2/3}y}\,\partial_y + \frac{W_0(y)}{\sqrt{1-\varepsilon^{2/3}y}}$$

and $F_{N,\varepsilon}(y,R) = F_{N,0}(y) + G_{N,\varepsilon}(y,R)$ with

$$\|F_{N,0}\|_{L^2_\varepsilon} \lesssim 1, \quad \|G_{N,\varepsilon}\|_{H^1_\varepsilon} \lesssim \varepsilon^{2/3} + \varepsilon^{(2N+1)/3}\|R\|^2_{H^1_\varepsilon} + \varepsilon^{4(N+1)/3}\|R\|^3_{H^1_\varepsilon}.$$

Here the norm in H^1_ε is defined by

$$\|u\|^2_{H^1_\varepsilon} := \int_{-\infty}^{\varepsilon^{-2/3}} \left[\frac{W_0(y)u(y)^2}{\sqrt{1-\varepsilon^{2/3}y}} + 4\sqrt{1-\varepsilon^{2/3}y}\,(u'(y))^2\right] dy$$

and we show that H^1_ε is a Banach algebra with Sobolev's embedding

$$\|u\|_{L^\infty(J_\varepsilon)} \leq C\|u\|_{H^1_\varepsilon},$$

where C is ε-independent.

Grand finale

- The map
$$\Psi_\varepsilon : f \mapsto \phi := (T^\varepsilon)^{-1} \frac{f}{\sqrt{1 - \varepsilon^{2/3} y}}$$
is continuous from L^2_ε into H^1_ε and the norm of Ψ_ε is uniformly bounded in ε.

- By the Fixed Point Theorem, there exists a unique fixed point $R_{N,\varepsilon} \in H^1_\varepsilon$ such that
$$\|R_{N,\varepsilon} - R^0_{N,\varepsilon}\|_{H^1_\varepsilon} \lesssim \varepsilon^{2/3} + \varepsilon^{(2N+1)/3}.$$

- We prove that $\nu_\varepsilon(y) > 0$ for all $y \in J_\varepsilon$ so that it is the ground state η_ε by uniqueness of the positive solution η_ε.

Operators of the linearized problem

The spectral problem is given by

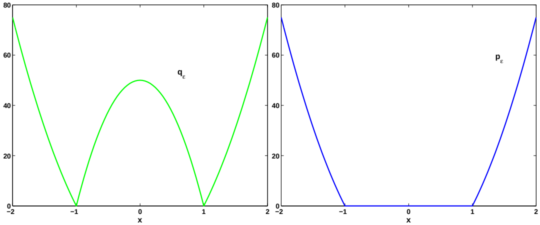
$$L_+^{\varepsilon} u = -\lambda w, \quad L_-^{\varepsilon} w = \lambda u,$$

where

$$L_+^{\varepsilon} = -\varepsilon^2 \partial_x^2 + V_\varepsilon(x), \quad V_\varepsilon(x) = 3\eta_\varepsilon^2(x) - 1 + x^2,$$

and

$$L_-^{\varepsilon} = -\varepsilon^2 \partial_x^2 + \tilde{V}_\varepsilon(x), \quad \tilde{V}_\varepsilon(x) = \eta_\varepsilon^2(x) - 1 + x^2 = \frac{\varepsilon^2 \eta_\varepsilon''(x)}{\eta_\varepsilon(x)}.$$



Semi-classical limit for eigenvalues of L_+^ε

Consider the eigenvalue problem

$$\left(-\partial_x^2 + \varepsilon^{-2} V_\varepsilon(x)\right) u_n(x) = \varepsilon^{-2} \lambda_n u_n(x), \quad x \in \mathbb{R},$$

where

- $V_\varepsilon(x) \in C^\infty(\mathbb{R})$ for any small $\varepsilon > 0$,
- $\lim_{\varepsilon \to 0} V_\varepsilon(x) = V_0(x) \in C(\mathbb{R})$ given by

$$V_0(x) = \begin{cases} 2(1-x^2), & |x| \leq 1, \\ x^2 - 1, & |x| \geq 1, \end{cases}$$

- $V_\varepsilon(x)$ takes its absolute minimum near $x = \pm 1$, and
- $V_\varepsilon(x) \to +\infty$ as $|x| \to \infty$.

By the Bohr–Sommerfeld rule,

$$\frac{1}{\pi} \int_{x_-^\varepsilon}^{x_+^\varepsilon} \sqrt{\lambda - V_\varepsilon(x)} dx \sim \varepsilon \left(n - \frac{1}{2}\right), \quad \text{as} \quad \varepsilon \to 0, \;\; n \geq 1,$$

Reduction to the linearized Painlevé equation

Changing variables

$$y = \frac{1 - x^2}{\varepsilon^{2/3}}, \quad \lambda = \varepsilon^{2/3} \mu, \quad V_+^{\varepsilon} = \varepsilon^{2/3} W_{\varepsilon}(y),$$

where $W_{\varepsilon}(y) = 3\nu_{\varepsilon}^2(y) - y$, we obtain

$$\int_{y_-^{\varepsilon}}^{y_+^{\varepsilon}} \frac{\sqrt{\mu - W_{\varepsilon}(y)}}{\sqrt{1 - \varepsilon^{2/3} y}} dy \sim 2\pi \left(n - \frac{1}{2} \right), \quad \text{as} \quad \varepsilon \to 0, \;\; n \geq 1.$$

Claim: The quantization formula above does not give a correct limit $\varepsilon \to 0$ at least for small $n \geq 1$. Instead, the eigenvalues $\{\mu_n^{\varepsilon}\}_{n \geq 1}$ converge to eigenvalues of the linearized Painlevé operator

$$M_0 u(y) := -4u''(y) + W_0(y) u(y) = \mu u(y).$$

Convergence of eigenvalues

Theorem

For $\varepsilon > 0$ sufficiently small, the spectrum of L_+^ε consists of an increasing sequence of positive eigenvalues $\{\lambda_n^\varepsilon\}_{n\geq 1}$ such that for each $n \geq 1$,

$$\lim_{\varepsilon \downarrow 0} \frac{\lambda_{2n-1}^\varepsilon}{\varepsilon^{2/3}} = \lim_{\varepsilon \downarrow 0} \frac{\lambda_{2n}^\varepsilon}{\varepsilon^{2/3}} = \mu_n. \tag{1}$$

Further news: The same results can be extended in the space of d dimensions for radially symmetric parabolic traps:

$$iu_t + \varepsilon^2 \Delta u + (1 - |\mathbf{x}|^2)u - |u|^2 u = 0, \quad \mathbf{x} \in \mathbb{R}^d,$$

for any $d \geq 1$.