Instabilities of steep Stokes waves

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joint work with S. Dyachenko (Buffalo), R. Marangell (Sydney)

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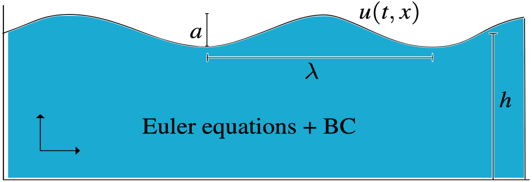
1. Stability patterns of the surface water waves

Stokes waves are traveling waves of the permanent form in the irrotational motion of an incompressible fluid:



Traveling waves of small amplitudes were defined in

G. G. Stokes, On the theory of oscillatory waves, Transactions of the Cambridge Philosophical Society 8 (1847) 441

A. I. Nekrasov. On waves of permanent type. Ivanovo Polite. Inst. 3 (1921) 52

T. Levi-Civita, Determination rigoureuse des ondes permanentes dámpleur finite, Mathematische Annalen 93 (1925) 264

D. Struik. D´etermination rigoureuse des ondes irrotationelles p´eriodiques dans un canal ¨a profondeur finie. Math. Ann. 95 (1926) 595

Recent numerical results on existence and stability

Numerical approximations of smooth periodic waves of large amplitudes were obtained with a high precision.

S. Dyachenko, P. Lushnikov, A. Korotkevich, Stud. Appl. Math. (2016)

S. Dyachenko, V. Hur, D. Silantyev, J. Fluid Mech. **955** (2023) A17

Spectral stability of smooth Stokes waves was explored numerically both for co-periodic and localized perturbations.

S. Dyachenko, A. Semenova, J. Comp. Phys. **492** (2023) 112411

A. Korotkevich, P. Lushnikov, A. Semenova, S. Dyachenko, Stud.Appl.Math.(2023)

B. Deconinck, S. Dyachenko, A. Semenova, J. Fluid Mech. **995** (2024) A2

B. Deconinck, S. Dyachenko, P. Lushnikov, A. Semenova, Proc.Nat.Acad.Sci. (2023)

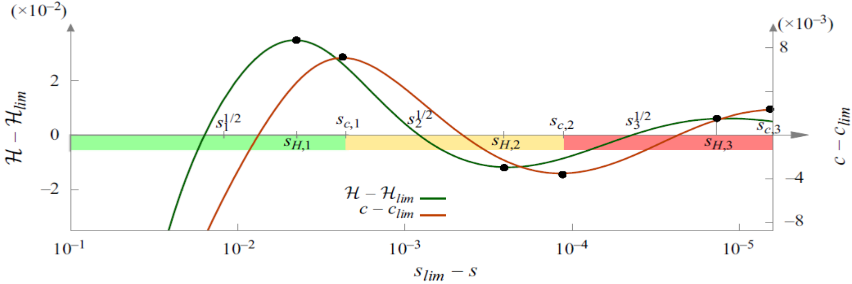
Recent results on subharmonic bifurcations and stability for the finite-depth fluids were obtained.

A. Semenova, Appl. Math. Lett. **167** (2025) 109560

E. Byrns, B. Deconinck, A. Semenova, J. Nonlinear Waves **2** (2026)

Recent numerical results on existence and stability

The wave energy $\mathcal{H}$ and the wave speed c oscillate as functions of the wave steepness s towards the limiting wave with the peaked profile:

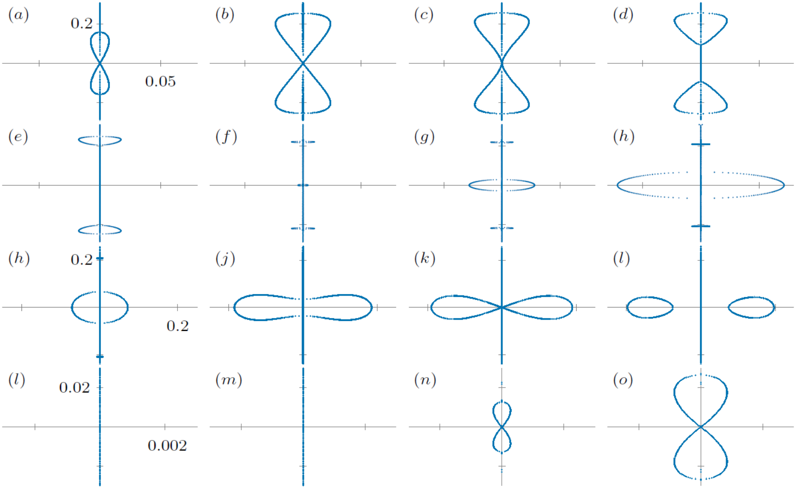


Conjectures based on the numerical results:

- $\exists$ ∞-many oscillations of wave energy and speed.
- Unstable eigenvalues appear past every extremal point of energy.

Recent numerical results on existence and stability

Instability bands with respect to arbitrary perturbations change in a sequence of bifurcations, which are repeated as steepness s increases.



Analytical results on stability and bifurcations

- Instability bifurcations at the extremal points of energy.

  M. Tanaka, J. Phys. Soc. Japan **52** (1983) 3047; J. Fluid Mech. **156** (1985) 281

  P. G. Saffman, J. Fluid Mech. **159** (1985) 169

- Global bifurcation theory
  - ∞-many extremal points of speed (co-periodic bifucations),
  - ∞-many points of subharmonic bifurcations.

  B. Buffoni, E. N. Dancer, J. F. Toland, ARMA **152** (2000) 207–271

- Our principal results on stability bifurcations

  S. Dyachenko, D.P., Physica D **483** (2025) 134925

  S. Dyachenko, D.P., R. Marangell, J. Noninear Waves (2026) submitted

2. Advantages of the pseudo-differential formulation

- $\eta(x,t)$ - the free surface profile with the zero-mean constraint

$$\oint \eta dx = 0.$$

- $\phi(x,y,t)$ - velocity potential satisfying the Laplace equation in

$$D_\eta(t) := \{(x,y): \quad -\pi \leq x \leq \pi, \quad -\infty < y \leq \eta(x,t)\}$$

- Periodic boundary conditions at $x = \pm\pi$.
- Decaying boundary condition $\varphi_y|_{y\to-\infty} = 0$.
- Nonlinear evolution equations at the free surface:

$$\left.\begin{aligned} \eta_t + \varphi_x\eta_x - \varphi_y = 0, \\ \varphi_t + \frac{1}{2}(\varphi_x)^2 + \frac{1}{2}(\varphi_y)^2 + \eta = 0, \end{aligned}\right\} \qquad \text{at } \; y = \eta(x,t),$$

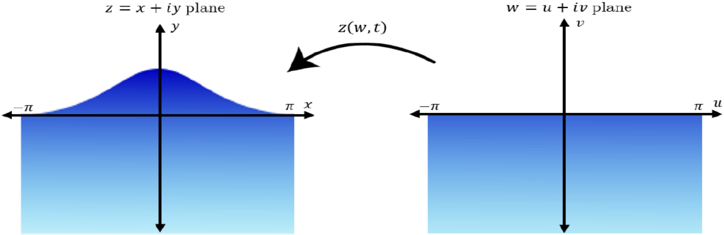
2. Advantages of the pseudo-differential formulation

Three methods of study the dynamics in the Hamiltonian system

$$\left. \begin{aligned} \eta_t + \varphi_x \eta_x - \varphi_y = 0, \\ \varphi_t + \frac{1}{2}(\varphi_x)^2 + \frac{1}{2}(\varphi_y)^2 + \eta = 0, \end{aligned} \right\} \qquad \text{at } \; y = \eta(x,t),$$

1. Dirichlet-to-Neumann operator for $\varphi_y|_{y=\eta(x,t)}$ by
   W. Craig, C. Sulem, J. Comput. Phys. 108 (1993) 73
2. Reformulation in conformal variables by
   A. I. Dyachenko, V. E. Zakharov, *et al.*, Phys. Lett. A 221 (1996) 73
3. Reformulation with integral constraints by
   M. Ablowitz, A. Fokas, Z. Musslimani, J. Fluid Mech. 562 (2006) 313

Conformal transformation

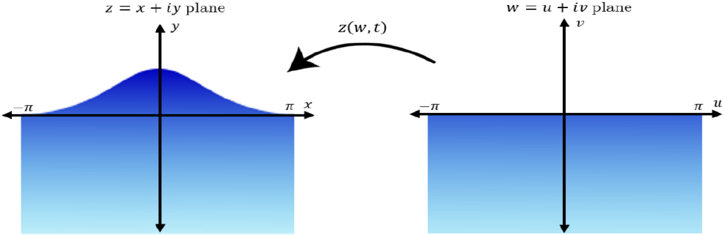


Cauchy–Riemann equations for $z = F(w,t)$ with holomorphic F in w:

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$$

in the fixed domain $\mathcal{D} := \{(u,v): \quad -\pi \leq u \leq \pi, \ -\infty \leq v \leq 0\}$ with decaying condition as $v \to -\infty$ and periodicity at $u = \pm\pi$.

Conformal transformation



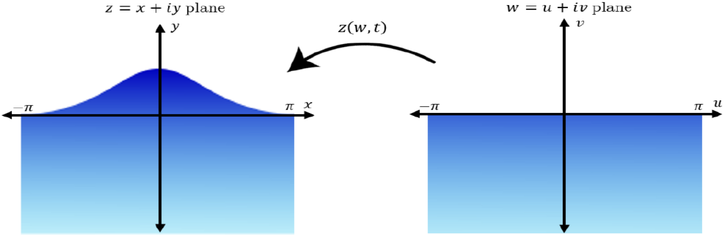
Fourier series solution:

$$x(u,v,t) = u + \sum_{n\in\mathbb{Z}} \hat{x}_n(t) e^{inu}\ e^{|n|v},$$

$$y(u,v,t) = v + \hat{y}_0(t) + \sum_{n\in\mathbb{Z}} \hat{x}_n(t) e^{inu}\ i\ \mathrm{sgn}(n) e^{|n|v},$$

where $\hat{y}_0(t)$ does not affect equations of motion.

Conformal transformation



Similarly, the velocity potential is uniquely represented by

$$\varphi(u, v, t) = \sum_{n \in \mathbb{Z}} \hat{\phi}_n(t) e^{inu} \ e^{|n|v},$$

where $\hat{\phi}_n(t)$ is the Fourier coefficient for $\phi(u,t) = \varphi(u, v = 0, t)$. The other variables are $\xi(u,t) = x(u, v = 0, t)$ and $\eta(u,t) = y(u, v = 0, t)$.

A closed system of evolution equations

It follows from Cauchy–Riemann equations for $z = F(w,t)$ that

$$\eta = \hat{y}_0 + \mathcal{H}(\xi - u),$$

where $\mathcal{H}$ is Hilbert transform with Fourier symbol $\widehat{\mathcal{H}}_n = i\mathrm{sgn}(n)$.

The closed system of evolution equations is obtained by the least action principle from the Lagrangian

$$\begin{aligned}\mathcal{L}(\phi,\xi,\eta) := \oint \phi(\eta_t\xi_u - \eta_u\xi_t)du - \frac{1}{2}\oint \phi\mathcal{K}\phi du - \frac{1}{2}\oint \eta^2\xi_u du \\ + \oint f(\eta - \hat{y}_0 - \mathcal{H}(\xi - u))du,\end{aligned}$$

where $\mathcal{K} = -\mathcal{H}\partial_u = |\partial_u|$ is a positive self-adjoint operator in $L^2(\mathbb{T})$ and f is the Lagrange multiplier to preserve the constraint.

A closed system of evolution equations

Using Euler–Lagrange equations in (ϕ, η, ξ), we obtain

$$\left\{ \begin{array}{l} \eta_t \xi_u - \eta_u \xi_t + \mathcal{H} \phi_u = 0, \\ -\phi_t \xi_u + \phi_u \xi_t - \eta \xi_u + f = 0, \\ \phi_t \eta_u - \phi_u \eta_t + \eta \eta_u + \mathcal{H} f = 0, \end{array} \right.$$

where $\xi_u = 1 + \mathcal{K} \eta$ and $\xi_t = \mathcal{H} \eta_t$.

A closed system of evolution equations

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$$\left\{ \begin{array}{l} \eta_t \xi_u - \eta_u \xi_t + \mathcal{H}\phi_u = 0, \\ -\phi_t \xi_u + \phi_u \xi_t - \eta \xi_u + f = 0, \\ \phi_t \eta_u - \phi_u \eta_t + \eta \eta_u + \mathcal{H} f = 0, \end{array} \right.$$

where $\xi_u = 1 + \mathcal{K}\eta$ and $\xi_t = \mathcal{H}\eta_t$.

Eliminating f yields the closed system of two evolution equations

$$\left\{ \begin{array}{l} (1 + \mathcal{K}\eta)\eta_t + \eta_u \mathcal{H}\eta_t + \mathcal{H}\phi_u = 0, \\ \phi_t \eta_u - \phi_u \eta_t + \eta \eta_u + \mathcal{H}\left[(1 + \mathcal{K}\eta)\phi_t + \phi_u \mathcal{H}\eta_t + (1 + \mathcal{K}\eta)\eta\right] = 0, \end{array} \right.$$

which replaces a full system of equations of motion for water waves.

The system admits a Hamiltonian formulation in canonical variables.

Conserved quantities

The mass conservation is $\oint \eta \xi_u du = \oint \eta(1+\mathcal{K}\eta) du = 0$, where the constraint is imposed due to the choice $\oint \eta dx = 0$.

Additional conserved quantities are two (vertical and horizontal) components of the momentum and the energy:

$$\oint \phi(1+\mathcal{K}\eta) du, \quad \oint \phi \eta_u du, \quad \oint \left[\eta^2(1+\mathcal{K}\eta) + \phi\mathcal{K}\phi\right] du.$$

No other conserved quantities exist in the closed system of equations, both in the original equations and in the conformal variables.

T. Benjamin, P. Olver, J. Fluid Mech. 125 (1982) 137

Babenko's equation

Traveling waves $\eta(u,t) = \eta(u - ct)$ satisfy $\phi = -c\mathcal{H}\eta$, where the profile η is a solution of a scalar pseudo–differential equation

$$\boxed{(c^2\mathcal{K} - 1)\eta = \frac{1}{2}\mathcal{K}\eta^2 + \eta\mathcal{K}\eta, \quad \mathcal{K} = -\mathcal{H}\partial_u = |\partial_u|.}$$

Existence of traveling waves with both smooth and peaked profiles is defined by solutions of Babenko's equation.

K. Babenko, Russian Academy of Sciences 294 (1987) 1033

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Existence of traveling waves with both smooth and peaked profiles is defined by solutions of Babenko's equation.

K. Babenko, Russian Academy of Sciences 294 (1987) 1033

Linear stability of traveling waves is defined by the linearization of the two equations near the traveling wave profile:

$$\left\{ \begin{array}{l} \eta(u,t) = \eta(u-ct) + v(u-ct,t), \\ \phi(u,t) = -c\mathcal{H}\eta(u-ct) + w(u-ct,t). \end{array} \right.$$

S. Dyachenko, A. Semenova, J. Comp. Phys. 492 (2023) 112411

Babenko's equation

Linearization at (v, w) yields the closed system of linear equations

$$\begin{cases} \mathcal{M} v_t & = \mathcal{K} w, \\ \mathcal{M}^* w_t - 2c\mathcal{H} v_t & = \mathcal{L} v, \end{cases}$$

where $\mathcal{M} = 1 + \mathcal{K}\eta + \eta' \mathcal{H}$, $\mathcal{M}^* = 1 + \mathcal{K}\eta - \mathcal{H}(\eta' \cdot)$, and

$$\mathcal{L} = c^2 \mathcal{K} - (1 + \mathcal{K}\eta) - \eta \mathcal{K} - \mathcal{K}(\eta \cdot),$$

where $\mathcal{L}$ is a self-adjoint linearized Babenko operator in $L^2(\mathbb{T})$.

Babenko's equation

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$$\mathcal{L} = c^2\mathcal{K} - (1 + \mathcal{K}\eta) - \eta\mathcal{K} - \mathcal{K}(\eta \cdot),$$

where $\mathcal{L}$ is a self-adjoint linearized Babenko operator in $L^2(\mathbb{T})$.

Separation of variables, e.g. $v(u - ct, t) = \hat{v}(u - ct)e^{\lambda t}$, yields the spectral stability problem

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

with the domain in $H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ for co-periodic perturbations.

3. Stability criterion for co-periodic perturbations

The traveling wave with profile $\eta \in C^\infty_{\rm per}(\mathbb{T})$ exists for $c \in (1, c_{\max})$.

Consider solutions $(\hat{v}, \hat{w}) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ of the spectral stability problem for some values of $\lambda \in \mathbb{C}$:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

where

- $\mathcal{H}$, $\mathcal{M}$, and $\mathcal{M}^*$ are bounded operators in $L^2(\mathbb{T})$.
- $\mathcal{K} = -\mathcal{H}\partial_u$ and $\mathcal{L}$ are unbounded operators in $L^2(\mathbb{T})$ with the domain in $H^1_{\rm per}(\mathbb{T})$. The spectrum of λ is purely discrete.
- We have $\mathcal{K}1 = 0$. Since $\mathcal{L}$ is the linearized Babenko's operator, we also have $\mathcal{L}\eta' = 0$ and $\mathcal{L}\partial_c \eta = -2c\mathcal{K}\eta$.
- The spectrum has Hamiltonian symmetry: λ, $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$.

What determine new unstable EV (eigenvalue) λ?

For the linearized Babenko's operator $\mathcal{L} : H^1_{\rm per}(\mathbb{T}) \to L^2(\mathbb{T})$, the spectrum is known at $c = 1$, where $\eta = 0$:

$$c = 1 : \quad \sigma(\mathcal{L}) = \{|n| - 1, \quad n \in \mathbb{Z}\} = \{-1, 0, 1, 2, \dots\}.$$

- Zero EV splits into a zero eigenvalue associated with the odd eigenfunction $\eta'(u)$ and a small negative EV associated with an even eigenfunction.

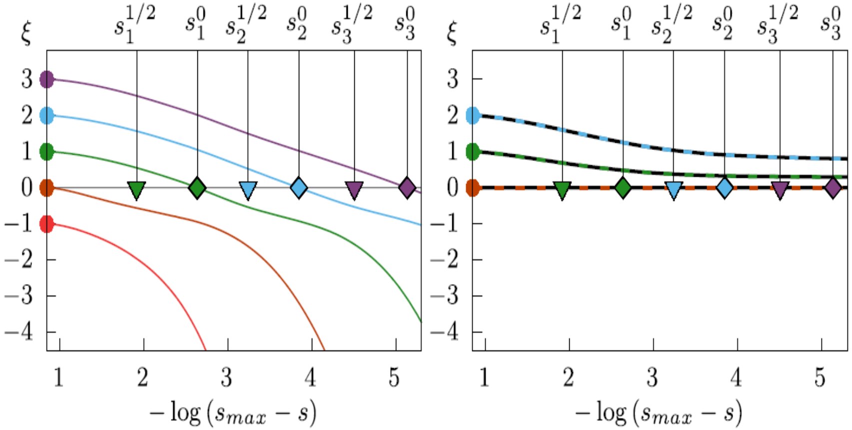
- Similarly, all double positive eigenvalues split for even and odd eigenfunctions. Eigenvalues for even eigenfunctions cross 0 for larger steepness. Eigenvalues for odd eigenfunctions do not cross 0 for larger steepness.

B. Buffoni, E. N. Dancer, J. F. Toland, ARMA **152** (2000) 207–271

What determine new unstable EV (eigenvalue) λ?

Numerical results from

S. Dyachenko, A. Semenova, J. Comp. Phys. 492 (2023) 112411



What determine new unstable EV (eigenvalue) λ?

Q: Do unstable eigenvalues bifurcate when the linearized Babenko's operator admit zero EV?

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

$$\begin{aligned} c \neq c_0 : &\quad \mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1), \quad \mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta'), \\ c = c_0 : &\quad \mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1), \quad \mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta', v_0), \end{aligned}$$

What determine new unstable EV (eigenvalue) λ?

Q: Do unstable eigenvalues bifurcate when the linearized Babenko's operator admit zero EV?

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

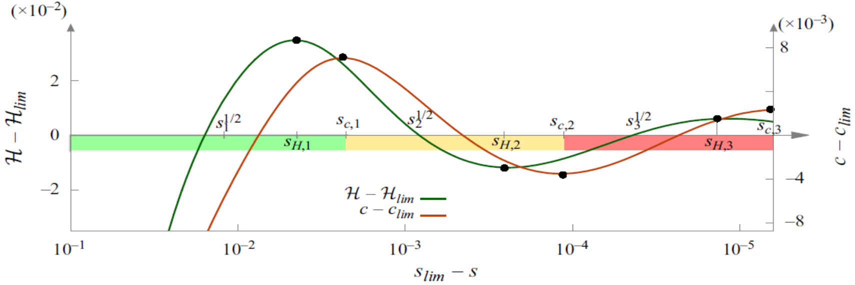
$$\begin{aligned} c \neq c_0: &\quad \mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1), \quad \mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta'), \\ c = c_0: &\quad \mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1), \quad \mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta', v_0), \end{aligned}$$

A: No. They bifurcate at the extremal points of energy, for which the algebraic multiplicity of the zero EV increases.

$$H(\eta) = \oint \left[\eta^2(1 + \mathcal{K}\eta) + c^2 \eta \mathcal{K}\eta\right] du.$$

P. G. Saffman, J. Fluid Mech. **159** (1985) 169

What determine new unstable EV (eigenvalue) λ?



Babenko's equation is the Euler–Lagrange equation of the action

$$\Lambda_c(\eta) := \frac{1}{2}\langle (c^2 K - 1)\eta, \eta\rangle - \frac{1}{2}\langle K\eta^2, \eta\rangle,$$

so that

$$\frac{d}{dc}\Lambda_c(\eta) = P(\eta), \quad \frac{d}{dc}H(\eta) = c\frac{d}{dc}P(\eta),$$

where $P(\eta) = c\langle \mathcal{K}\eta, \eta\rangle$ is the horizontal momentum.

Bifurcations of unstable eigenvalues

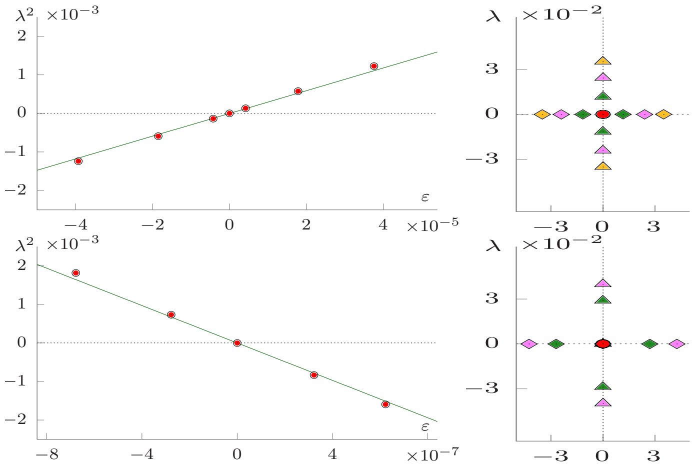
Theorem (S. Dyachenko–D. P, Physica D (2025))

Assume $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$ *and* $\mathcal{B} \neq 0$. *The generalized null space is four-dimensional if* $\frac{d}{dc}P(\eta) \neq 0$ *and six-dimensional if* $\frac{d}{dc}P(\eta) = 0$. *Moreover, new eigenvalues* λ *satisfy*

$$\mathcal{B}\lambda^2 + (c - c_0)\frac{d^2}{dc^2}P(\eta)|_{c=c_0} + \mathcal{O}((c-c_0)^2) = 0.$$

- The numerical coefficient $\mathcal{B} > 0$ is approximated numerically.
- Since $c - c_0$ and $\frac{d^2}{dc^2}P(\eta)|_{c=c_0}$ alternate between each extremal point, the new unstable eigenvalues always bifurcate in the direction of increasing steepness.

Bifurcations of unstable eigenvalues



Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

By the assumption, $\mathrm{Ker}(\mathcal{K}) = \mathrm{span}(1)$ and $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$. For every $\lambda \neq 0$, we have two orthogonality conditions:

$$\langle 1, \mathcal{M}v \rangle = \langle 1 + 2\mathcal{K}\eta, v \rangle = 0,$$
$$\langle \eta', -2c\mathcal{H}v + \mathcal{M}^* w \rangle = -2c\langle \mathcal{K}\eta, v \rangle + \langle \eta', w \rangle = 0.$$

These constraints determine the size of the Jordan block at $\lambda = 0$.

Jordan block computations

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$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

Kernel at $\lambda = 0$:

$$\left(\begin{array}{c} v \\ w \end{array} \right) = a_1 \left(\begin{array}{c} \eta' \\ 0 \end{array} \right) + a_2 \left(\begin{array}{c} 0 \\ 1 \end{array} \right),$$

where (a_1, a_2) are arbitrary.

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

First generalized kernel at $\lambda = 0$:

$$\left\{ \begin{array}{ll} \mathcal{K} w_1 & = a_1 \mathcal{M} \eta', \\ \mathcal{L} v_1 & = -2c a_1 \mathcal{H} \eta' + a_2 \mathcal{M}^* 1. \end{array} \right.$$

There exists a solution $(v_1, w_1) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$:

$$\left(\begin{array}{c} v \\ w \end{array} \right) = a_1 \left(\begin{array}{c} -\partial_c \eta \\ \mathcal{H} \eta \end{array} \right) + a_2 \left(\begin{array}{c} -1 \\ 0 \end{array} \right).$$

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

Second generalized kernel at $\lambda = 0$:

$$\left\{ \begin{array}{ll} \mathcal{K}w_2 & = -a_1 \mathcal{M} \partial_c \eta - a_2 \mathcal{M} 1, \\ \mathcal{L}v_2 & = 2ca_1 \mathcal{H} \partial_c \eta + a_1 \mathcal{M}^* \mathcal{H} \eta. \end{array} \right.$$

For the first equation, $\langle 1, \mathcal{M}\partial_c \eta \rangle = 0$ but $\langle 1, \mathcal{M} 1 \rangle = 1 \neq 0$.

For the second equation, $\langle \eta', 2c\mathcal{H}\partial_c \eta + \mathcal{M}^* \mathcal{H} \eta \rangle = \frac{d}{dc} P(\eta)$.

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

The Jordan canonical forms are

$$\frac{d}{dc} P(\eta) \neq 0 : \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\frac{d}{dc} P(\eta) = 0, \quad \mathcal{B} \neq 0 : \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Jordan block computations

We have the spectral stability problem:

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

If $\frac{d}{dc}P(\eta) = 0$, the Jordan chain of eigenvectors for $(a_1, a_2) = (1, 0)$ is

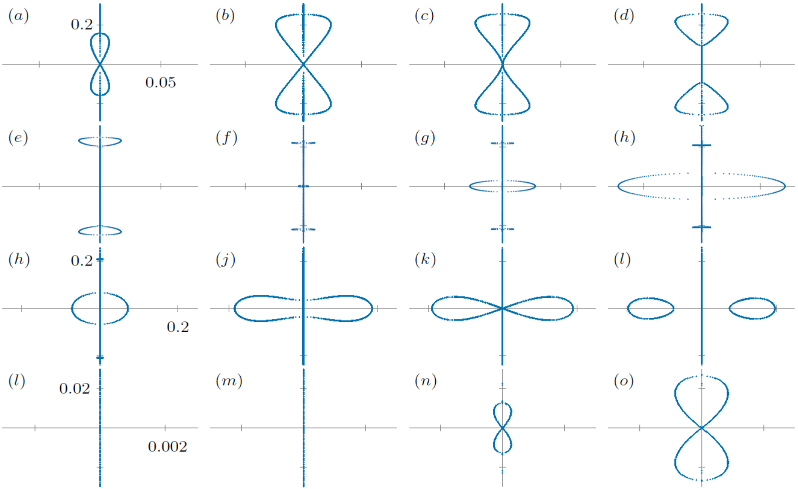
$$\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} \eta' \\ 0 \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} -\partial_c \eta \\ \mathcal{H}\eta \end{pmatrix}, \quad \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}, \quad \begin{pmatrix} v_3 \\ w_3 \end{pmatrix},$$

where

$$\left\{ \begin{array}{ll} \mathcal{K}w_2 & = -\mathcal{M}\partial_c \eta, \\ \mathcal{L}v_2 & = 2c\mathcal{H}\partial_c\eta + \mathcal{M}^*\mathcal{H}\eta. \end{array} \right. \qquad \left\{ \begin{array}{ll} \mathcal{K}w_3 & = \mathcal{M}v_2, \\ \mathcal{L}v_3 & = -2c\mathcal{H}v_2 + \mathcal{M}^* w_2. \end{array} \right.$$

and $\mathcal{B} := \langle \eta', -2c\mathcal{H}v_3 + \mathcal{M}^* w_3 \rangle$.
Justification of the normal form is done by Puiseux expansions.

4. Bifurcations of the modulational stability bands



B. Deconinck, S. Dyachenko, A. Semenova, J. Fluid Mech. 995 (2024) A2

Stability with respect to decaying perturbations

We consider solutions $(v, w) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ of

$$\begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{K} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H} & \mathcal{M}^* \\ \mathcal{M} & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

By the Floquet–Bloch theory, we can use the integral transform, e.g.

$$v(u) = \int_{\mathbb{B}} \hat{v}(u, \mu) e^{i\mu u} d\mu, \quad \mathbb{B} = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

and consider solutions $(\hat{v}, \hat{w}) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ of

$$\begin{pmatrix} \mathcal{L}_\mu & 0 \\ 0 & \mathcal{K}_\mu \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix} = \lambda \begin{pmatrix} -2c\mathcal{H}_\mu & \mathcal{M}^*_\mu \\ \mathcal{M}_\mu & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix},$$

where μ-dependent operators, e.g. $\mathcal{L}_\mu = e^{-i\mu u} \mathcal{L} e^{i\mu u}$.
The spectrum is still purely discrete for every $\mu \in \mathbb{B}$ and has the Hamiltonian symmetry: $\lambda(\mu)$, $-\bar{\lambda}(\mu)$, $\bar{\lambda}(-\mu)$, $-\lambda(-\mu)$.

Stability with respect to decaying perturbations

$\underline{\mathcal{H}_\mu = e^{-i\mu x}\mathcal{H}e^{i\mu x}:}$

Since $\mathcal{H}\sum_{n\in\mathbb{Z}}\hat{f}_n e^{i(\mu+n)u} = i\sum_{n\in\mathbb{Z}}\mathrm{sgn}(\mu+n)\hat{f}_n e^{i(\mu+n)u}$, we get

$$\mathcal{H}_\mu f = \mathcal{H}f \pm i\langle 1, f\rangle =: \mathcal{H}_\pm, \quad \pm\mu > 0.$$

Stability with respect to decaying perturbations

$\underline{\mathcal{H}_\mu = e^{-i\mu x}\mathcal{H}e^{i\mu x}:}$

Since $\mathcal{H}\sum_{n\in\mathbb{Z}}\hat{f}_n e^{i(\mu+n)u} = i\sum_{n\in\mathbb{Z}}\mathrm{sgn}(\mu+n)\hat{f}_n e^{i(\mu+n)u}$, we get

$$\mathcal{H}_\mu f = \mathcal{H}f \pm i\langle 1, f\rangle =: \mathcal{H}_\pm, \quad \pm\mu > 0.$$

$\underline{\mathcal{K}_\mu = e^{-i\mu x}\mathcal{K}e^{i\mu x}:}$

Since $\mathcal{K}_\mu = -\mathcal{H}_\mu(\partial_u + i\mu)$, we get

$$\mathcal{K}_\mu = \mathcal{K} - i\mu\mathcal{H}_\pm.$$

Stability with respect to decaying perturbations

$\underline{\mathcal{H}_\mu = e^{-i\mu x}\mathcal{H}e^{i\mu x}}$:
Since $\mathcal{H}\sum_{n\in\mathbb{Z}}\hat{f}_n e^{i(\mu+n)u} = i\sum_{n\in\mathbb{Z}}\mathrm{sgn}(\mu+n)\hat{f}_n e^{i(\mu+n)u}$, we get

$$\mathcal{H}_\mu f = \mathcal{H}f \pm i\langle 1, f\rangle =: \mathcal{H}_\pm, \quad \pm\mu > 0.$$

$\underline{\mathcal{K}_\mu = e^{-i\mu x}\mathcal{K}e^{i\mu x}}$:
Since $\mathcal{K}_\mu = -\mathcal{H}_\mu(\partial_u + i\mu)$, we get

$$\mathcal{K}_\mu = \mathcal{K} - i\mu\mathcal{H}_\pm.$$

$\underline{\mathcal{M}_\mu = e^{-i\mu x}\mathcal{M}e^{i\mu x}}$:
Similarly,

$$\begin{aligned}\mathcal{M}_\mu &= 1 + \mathcal{K}\eta + \eta'\mathcal{H}_\pm,\\ \mathcal{M}_\mu^* &= 1 + \mathcal{K}\eta - \mathcal{H}_\pm(\eta'\cdot).\end{aligned}$$

Stability with respect to decaying perturbations

$\underline{\mathcal{L}_\mu = e^{-i\mu x}\mathcal{L}e^{i\mu x}:}$
Since $\mathcal{L} = c^2\mathcal{K} - (1+\mathcal{K}\eta) - \eta\mathcal{K} - \mathcal{K}(\eta\ \cdot)$, we also have

$$\mathcal{L}_\mu = \mathcal{L} - i\mu\mathcal{L}_\pm,$$

where

$$\mathcal{L}_\pm = c^2\mathcal{H}_\pm - \eta\mathcal{H}_\pm - \mathcal{H}_\pm(\eta\cdot).$$

Hence, the spectral stability problem becomes a eigenvalue problem in λ with linear terms in μ:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}), \end{array} \right.$$

where $\pm\mu > 0$. Both problems are analytic in μ even at $\mu = 0$!

Stability with respect to decaying perturbations

In the limit $\mu \to 0^{\pm}$, we get the two limiting problems:

$$\left\{ \begin{array}{ll} \mathcal{K}\hat{w} & = \lambda \mathcal{M}_{\pm} \hat{v}, \\ \mathcal{L}\hat{v} & = \lambda (\mathcal{M}_{\pm}^* \hat{w} - 2c\mathcal{H}_{\pm}\hat{v}), \end{array} \right.$$

which is different from the co-periodic stability problem:

$$\left\{ \begin{array}{ll} \mathcal{K}w & = \lambda \mathcal{M} v, \\ \mathcal{L}v & = \lambda \left(\mathcal{M}^* w - 2c\mathcal{H}v \right). \end{array} \right.$$

Lemma: The limiting problems $\mu \to 0^{\pm}$ and the co-periodic stability problem are equivalent by the transformation:

$$\hat{v} = \mp i \langle 1, v \rangle \eta' + v, \qquad \hat{w} = \pm ic \langle 1, v \rangle + w.$$

Figure$-\infty$ bifurcations

Theorem (S. Dyachenko–R. Marangell–D. P (2026))

Assume $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$, $\mathcal{B} \neq 0$, *and* $\frac{d}{dc}P(\eta)|_{c=c_0} = 0$. *For* $c = c_0$, *there exist* **six** *spectral bands intersecting* $\lambda = 0$. *For* c *near* c_0, **two** *spectral bands are located near* $\lambda = 0$ *away from* $i\mathbb{R}$ *according to*

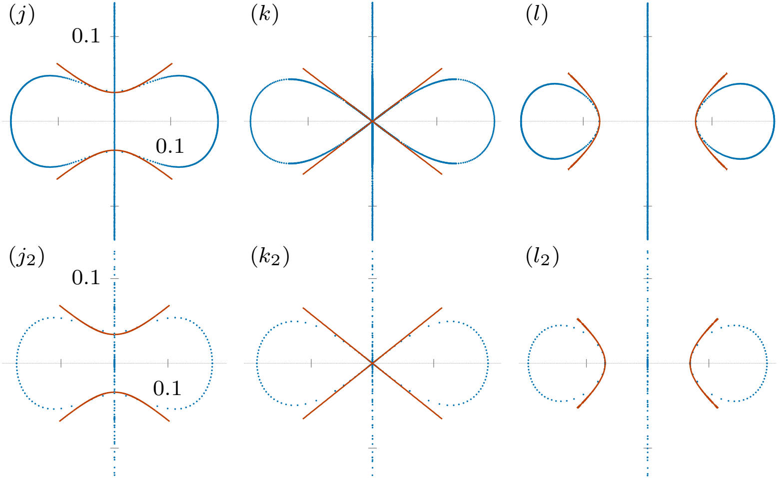
$$\lambda^2 \left(\mathcal{B}\lambda^2 + (c - c_0)\frac{d^2}{dc^2}P(\eta)|_{c=c_0} \right) - i\mu\lambda\frac{d}{dc}\|\eta\|^2_{L^2}|_{c=c_0} = \mathcal{O}(\mu^2).$$

- We proved that

$$c\frac{d}{dc}P(\eta) - \frac{d}{dc}\|\eta\|^2_{L^2} = 5P(\eta) \quad \Rightarrow \quad \frac{d}{dc}\|\eta\|^2_{L^2} < 0 \text{ if } \frac{d}{dc}P(\eta) = 0.$$

Figure$-\infty$ bifurcations

For two consequent bifurcations, the theory is triumphed as:



Jordan blocks and Newton polytopes

We study the spectral stability problem with $\pm\mu > 0$:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{array} \right.$$

The Jordan block structure at $(\mu, \lambda) = (0, 0)$:

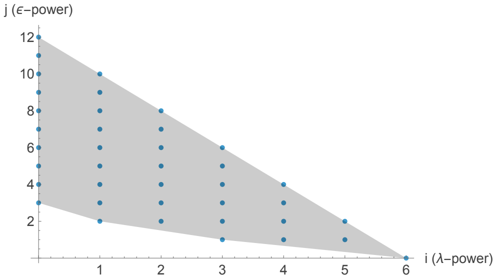
$$\frac{d}{dc}P(\eta) = 0, \quad \mathcal{B} \neq 0 : \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Jordan blocks and Newton polytopes

We study the spectral stability problem with $\pm\mu > 0$:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{cases}$$

Nonzero coefficients of the characteristic polynomial are shown by blue dots: $cp(\lambda,\mu) = \lambda^6 + a_3\mu\lambda^3 + a_4\mu^2\lambda + a_5\mu^3 + \text{h.o.t.}$ The lower convex hull is shown in grey: $(0,3)$, $(1,2)$, $(3,1)$, $(6,0)$.



Jordan blocks and Newton polytopes

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Newton's polytope at $(\mu, \lambda) = (0, 0)$ suggests the consistent scaling

$$\mu = \epsilon^3 \mu_1, \quad \lambda = \epsilon \lambda_1, \quad c - c_0 = \epsilon^2 c_1,$$

where $\mu_1 \neq 0$ and c_1 are given, and λ_1 is to be found with formal small parameter ϵ.

Jordan blocks and Newton polytopes

We study the spectral stability problem with $\pm\mu > 0$:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{array} \right.$$

The asymptotic expansion of solutions $(\hat{v}, \hat{w}) \in H^1_{\rm per}(\mathbb{T}) \times H^1_{\rm per}(\mathbb{T})$ is

$$\left(\begin{array}{c} \hat{v} \\ \hat{w} \end{array} \right) = \left(\begin{array}{c} \eta' \\ 0 \end{array} \right) + \epsilon \left(\begin{array}{c} \hat{v}_1 \\ \hat{w}_1 \end{array} \right) + \epsilon^2 \left(\begin{array}{c} \hat{v}_2 \\ \hat{w}_2 \end{array} \right) + \epsilon^3 \left(\begin{array}{c} \hat{v}_3 \\ \hat{w}_3 \end{array} \right) + \epsilon^4 \left(\begin{array}{c} \hat{v}_4 \\ \hat{w}_4 \end{array} \right) + \mathcal{O}(\epsilon^5),$$

We study the spectral stability problem with $\pm\mu > 0$:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda \mathcal{M}_\pm \hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^* \hat{w} - 2c\mathcal{H}_\pm \hat{v}). \end{cases}$$

First generalized kernel with the additional projection terms:

$$\left(\begin{array}{c} \hat{v}_1 \\ \hat{w}_1 \end{array} \right) = \lambda_1 \left[\left(\begin{array}{c} -\partial_c \eta \\ \mathcal{H}\eta \end{array} \right) \pm i \langle 1, \partial_c \eta \rangle \left(\begin{array}{c} \eta' \\ -c \end{array} \right) \right].$$

Jordan blocks and Newton polytopes

We study the spectral stability problem with $\pm\mu > 0$:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{array} \right.$$

Second generalized kernel at $\lambda = 0$:

$$\left(\begin{array}{c} \hat{v}_2 \\ \hat{w}_2 \end{array} \right) = \lambda_1^2 \left(\begin{array}{c} v_2 \\ w_2 \end{array} \right) + a \left(\begin{array}{c} 0 \\ 1 \end{array} \right).$$

where $a \in \mathbb{R}$ is to be determined.

Jordan blocks and Newton polytopes

We study the spectral stability problem with $\pm\mu > 0$:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{array} \right.$$

Third generalized kernel at $\lambda = 0$:

$$\left(\begin{array}{c} \hat{v}_3 \\ \hat{w}_3 \end{array} \right) = \lambda_1^3 \left[\left(\begin{array}{c} v_3 \\ w_3 \end{array} \right) \mp i\langle 1, v_3\rangle \left(\begin{array}{c} \eta' \\ -c \end{array} \right) \right] + i\mu_1 \left(\begin{array}{c} \frac{c}{2}\partial_c\eta - \eta \\ 0 \end{array} \right)$$
$$- a\lambda_1 \left(\begin{array}{c} 1 \\ 0 \end{array} \right).$$

Jordan blocks and Newton polytopes

We study the spectral stability problem with $\pm\mu > 0$:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{cases}$$

At the order of $\mathcal{O}(\epsilon^4)$, we get

$$\begin{cases} \mathcal{K}\hat{w}_4 & = \lambda_1\mathcal{M}_\pm\hat{v}_3 + i\mu_1\mathcal{H}_\pm\hat{w}_1, \\ \mathcal{L}\hat{v}_4 & = \lambda_1\left(\mathcal{M}_\pm^*\hat{w}_3 - 2c\mathcal{H}_\pm\hat{v}_3\right) + i\mu_1\mathcal{L}_\pm\hat{v}_1. \end{cases}$$

Fredholm condition for $\mathrm{Ker}(\mathcal{K})$ yields $a = \lambda_1^2\langle(1 + 2\mathcal{K}\eta), v_3\rangle$.
Fredholm condition for $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$ yields

$$\mathcal{B}\lambda_1^4 + c_1\lambda_1^2\frac{d^2}{dc^2}P(\eta)|_{c=c_0} - i\mu_1\lambda_1\frac{d}{dc}\|\eta\|_{L^2}^2 = 0.$$

Hourglass (vertical slope) bifucation

Theorem (S. Dyachenko–R. Marangell–D. P (2026))

Assume $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta')$ *and* $\frac{d}{dc}P(\eta) \neq 0$. *Let*

$$\Delta(c) := 4\mathcal{P}'(c)\mathcal{N}(c) - 5\mathcal{P}(c)\mathcal{N}'(c),$$

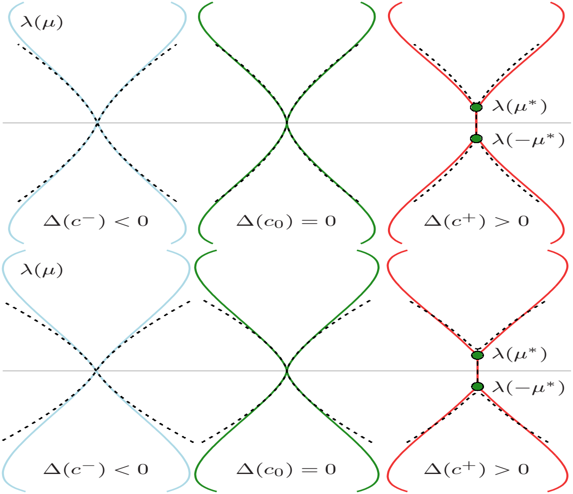
where $\mathcal{P}(c) = P(\eta)$ *and* $\mathcal{N}(c) = \|\eta\|^2_{L^2}$. *There exist* **four** *spectral bands intersecting* $\lambda = 0$. *If* $\Delta(c_0) = 0$ *for some* c_0, *then* **two** *spectral bands are tangential to* $i\mathbb{R}$ *and split for* $c \neq c_0$ *according to*

$$\mathcal{P}'(c_0)\left(\lambda - \frac{i\mu\mathcal{N}'(c)}{2\mathcal{P}'(c)}\right)^2 + \frac{\mu^2\Delta'(c_0)(c-c_0)}{4\mathcal{P}'(c_0)} + \mathcal{D}|\mu|^3 = \mathcal{O}(\mu^4),$$

where $\mathcal{D} \neq 0$ *and* $\Delta'(c_0) \neq 0$ *are assumed.*

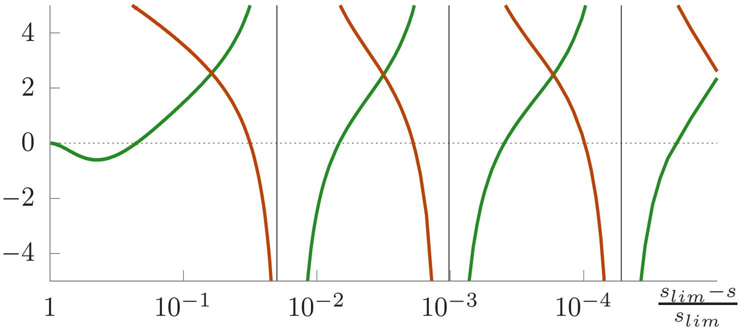
Hourglass (vertical slope) bifucation

For two consequent bifurcations, the theory is triumphed as:



Hourglass (vertical slope) bifucation

The dependence of $\mathcal{H}'(c)$ (red solid) and $\Delta(c)$ (green solid) versus the wave steepness s, with the vertical lines for the fold bifurcations (the double kernel of $\mathcal{L}$):



Hourglass (vertical slope) bifucation

We study the spectral stability problem with $\pm\mu > 0$:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{array} \right.$$

The Jordan block structure at $(\mu, \lambda) = (0, 0)$:

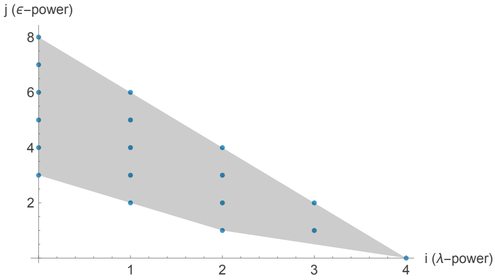
$$\frac{d}{dc}P(\eta) \neq 0 : \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hourglass (vertical slope) bifucation

We study the spectral stability problem with $\pm\mu > 0$:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{cases}$$

Nonzero coefficients of the characteristic polynomial are shown by blue dots: $cp(\lambda,\mu) = \lambda^4 + a_1\mu\lambda^2 + a_3\mu^2\lambda + a_4\mu^3 +$ h.o.t. The lower convex hull is shown in grey: $(0,3), (1,2), (2,1), (4,0)$.



Hourglass (vertical slope) bifucation

We study the spectral stability problem with $\pm\mu > 0$:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}). \end{cases}$$

Newton's polytope at $(\mu, \lambda) = (0, 0)$ suggests the consistent scaling

$$\mu = \epsilon\mu_1, \quad \lambda = \epsilon\lambda_1 + \epsilon^{3/2}\lambda_{3/2} + \mathcal{O}(\epsilon^2), \quad c - c_0 = \epsilon c_1,$$

where $\mu_1 \neq 0$, c_1 are given and λ_1 and $\lambda_{3/2}$ are to be found. The correction $\mathcal{O}(\epsilon^{3/2})$ arises if $\Delta(c_0) = 0$ for some c_0.

Figure-8 bifucation

Theorem (S. Dyachenko–R. Marangell–D. P (2026))

Assume $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta', v_0)$ *with* $v_0 \in H^1_{\rm per}$ *satisfying* $\langle 1, v_0 \rangle \neq 0$ *and* $\langle \mathcal{K} v_0, v_0^2 \rangle \neq 0$ *for this value of* $c = c_0$*. Assume that* $\mathcal{N}(c_0) - \frac{5c_0}{4}\mathcal{P}(c_0) < 0$*. There exist* **four** *spectral bands intersecting* $\lambda = 0$*, with* **two** *bands splitting as the figure-8 at one side of* $c = c_0$ *according to*

$$\left(\lambda - \frac{i\mu c_0}{2}\right)^2 = \mu^2 \left[\mathcal{N}(c_0) - \frac{5c_0}{4}\mathcal{P}(c_0)\right] \left[|\mu| + \frac{\delta_0 \varepsilon}{c_0^2 \langle 1, v_0 \rangle}\right] + \mathcal{O}(|\mu|^{7/2}),$$

where $c^2 = c_0^2 + \delta_0 \varepsilon^2 + \mathcal{O}(\varepsilon^4)$ *with*

$$\delta_0 = \frac{3}{2} \frac{\langle \mathcal{K} v_0, v_0^2 \rangle}{\langle \mathcal{K} \eta_0, v_0 \rangle}$$

and $\varepsilon = \mathcal{O}(|\mu|)$*.*

Figure-8 bifucation

For the second figure-8 bifurcation, the theory is triumphed as:

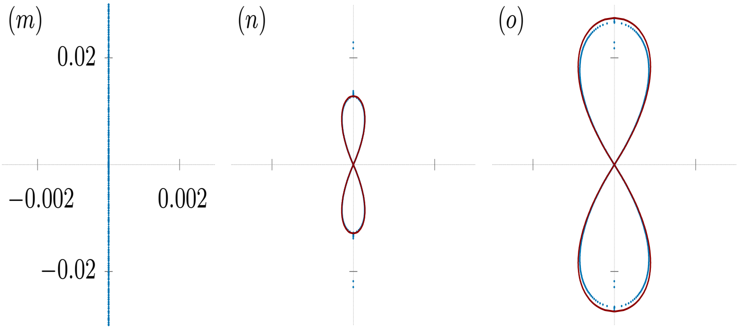


Figure-8 bifucation

For the first figure-8 bifurcation, the normal form is different since corresponds to the bifurcation of the Stokes wave itself at $c = 1$:

$$\left(\lambda - \frac{i\mu}{2}\right)^2 = \frac{1}{64}\mu^2(8a^2 - \mu^2) + \mathcal{O}(a^6 + \mu^6),$$

where $c = 1 + a^2 + \mathcal{O}(a^4)$.

M. Bert, A. Maspero, P. Ventura, Invent. Math. **230** (2022) 137–185

R. Creedon, B. Deconinck, J. Fluid Mech. **956** (2023) A29

Figure-8 bifucation

We study the spectral stability problem with $\pm\mu > 0$:

$$\left\{ \begin{array}{ll} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}) \end{array} \right.$$

The Jordan block structure at $(\mu, \lambda) = (0, 0)$:

$$\mathrm{Ker}(\mathcal{L}) = \mathrm{span}(\eta', v_0) : \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Figure-8 bifucation

We study the spectral stability problem with $\pm\mu > 0$:

$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_\pm)\hat{w} & = \lambda\mathcal{M}_\pm\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_\pm)\hat{v} & = \lambda(\mathcal{M}_\pm^*\hat{w} - 2c\mathcal{H}_\pm\hat{v}) \end{cases}$$

Nonzero coefficients of the characteristic polynomial are shown by blue dots: $cp(\lambda,\mu) = \lambda^4 + a_1\mu\lambda^2 + a_3\mu^2\lambda + a_4\mu^3 + \text{h.o.t.}$ The lower convex hull is shown in grey: $(0,3), (1,2), (2,1), (4,0)$.

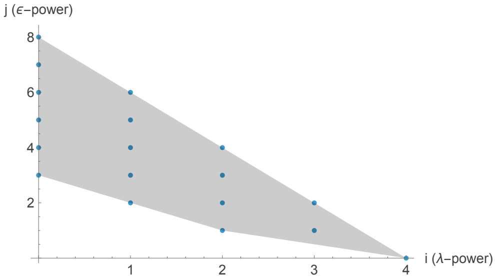


Figure-8 bifucation

We study the spectral stability problem with $\pm\mu > 0$:

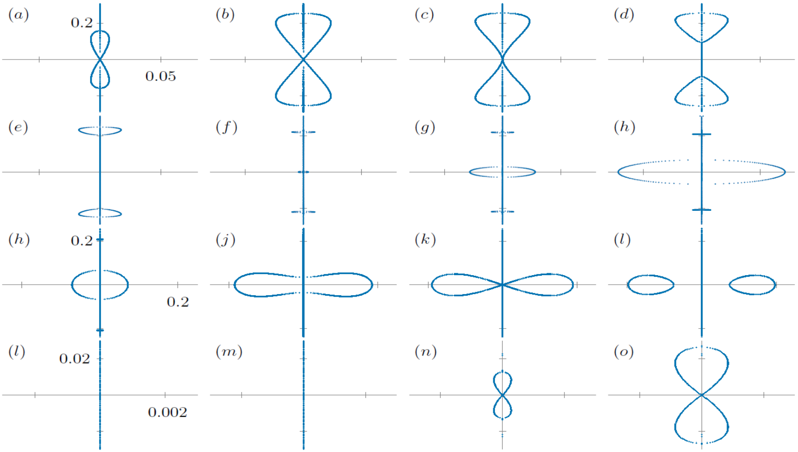
$$\begin{cases} (\mathcal{K} - i\mu\mathcal{H}_{\pm})\hat{w} & = \lambda\mathcal{M}_{\pm}\hat{v}, \\ (\mathcal{L} - i\mu\mathcal{L}_{\pm})\hat{v} & = \lambda(\mathcal{M}_{\pm}^{*}\hat{w} - 2c\mathcal{H}_{\pm}\hat{v}) \end{cases}$$

Newton's polytope at $(\mu, \lambda) = (0, 0)$ suggests the consistent scaling

$$\mu = \epsilon\mu_1, \quad \lambda = \epsilon\lambda_1 + \epsilon^{3/2}\lambda_{3/2} + \mathcal{O}(\epsilon^2), \quad c - c_0 = \epsilon^2 c_1,$$

where $\mu_1 \neq 0$, c_1 are given and λ_1 and $\lambda_{3/2}$ are to be found. The correction $\mathcal{O}(\epsilon^{3/2})$ arises because of the double degeneracy of $\lambda_1 = \frac{ic_0}{2}$.

This analysis gives three of the four recurrent bifurcations from $(\mu, \lambda) = (0, 0)$. The last one is a bifurcation from $(\mu, \lambda) = (\frac{1}{2}, 0)$ which corresponds to the double-periodic (subharmonic) bifurcation.



Summary

- The full water wave equations are replaced by two evolution equations via the conformal transformation.
- Existence of traveling waves is known from the global bifurcation analysis of the scalar pseudo–diferential equation.
- Stability of traveling waves in co-periodic perturbation is obtained from a purely discrete spectrum of the spectral problem.
- Bifurcations of unstable bands in the modulational stability problem at $\lambda = 0$ are analyzed with the Jordan block structure and the Newton polytope method.

The end!