Stationary and moving gap solitons in periodic potentials

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References:

Applicable Analysis, **86**, 1017-1036 (2007)

Mathematical Methods for Physical Sciences, submitted (2007)

Motivations

Examples:

Complex-valued Maxwell equation

$$\nabla^2 E - \left(1 + V(x) + \sigma |E|^2\right) E_{tt} = 0$$

and the Gross–Pitaevskii equation

$$iE_t = -\nabla^2 E + V(x)E + \sigma |E|^2 E,$$

where $E(x,t) : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{C}$,
$V(x) = V(x + 2\pi e_j) : \mathbb{R}^N \mapsto \mathbb{R}$,
and $\sigma = \pm 1$.

Gap solitons are localized stationary solutions of nonlinear PDEs with space-periodic coefficients which reside in a spectral gap of the associated linear Schrödinger operator.

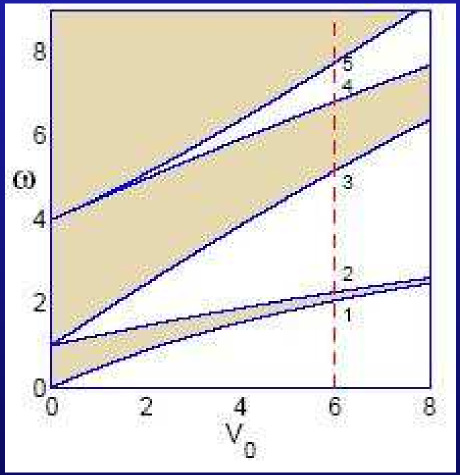
Existence of stationary solutions

Stationary solutions $E(x,t) = U(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a nonlinear elliptic problem with a periodic potential

$$\omega U = -\nabla^2 U + V(x)U + \sigma |U|^2 U$$

The associated Schrödinger equation in 1D is

$$\begin{cases} -u''(x) + V(x)u(x) = \omega u(x), \\ u(2\pi) = e^{i2\pi k}u(0), \end{cases}$$



Existence results

Previous results:

- Construction of multi-humped gap solitons in Alama-Li (1992)
- Bifurcations of gap solitons from band edges in Kupper-Stuart (1990) and Heinz-Stuart (1992)
- Multiplicity of branches of gap solitons in Heinz (1995)
- Existence of critical points of energy with L^2-normalization in Buffoni-Esteban-Sere (2006)

Theorem:[Pankov, 2005] Let $V(x)$ be a real-valued bounded periodic potential. Let ω be in a finite gap of the spectrum of $L = -\nabla^2 + V(x)$. There exists a non-trivial weak solution $U(x) \in H^1(\mathbb{R}^N)$, which is continuous on $x \in \mathbb{R}^N$ and decays exponentially as $|x| \to \infty$.

Illustration of solution branches

D.P., A. Sukhorukov, Yu. Kivshar, PRE **70**, 036618 (2004)
$V(x) = V_0 \sin^2(x)$ with $V_0 = 1$ and $\sigma = -1$:

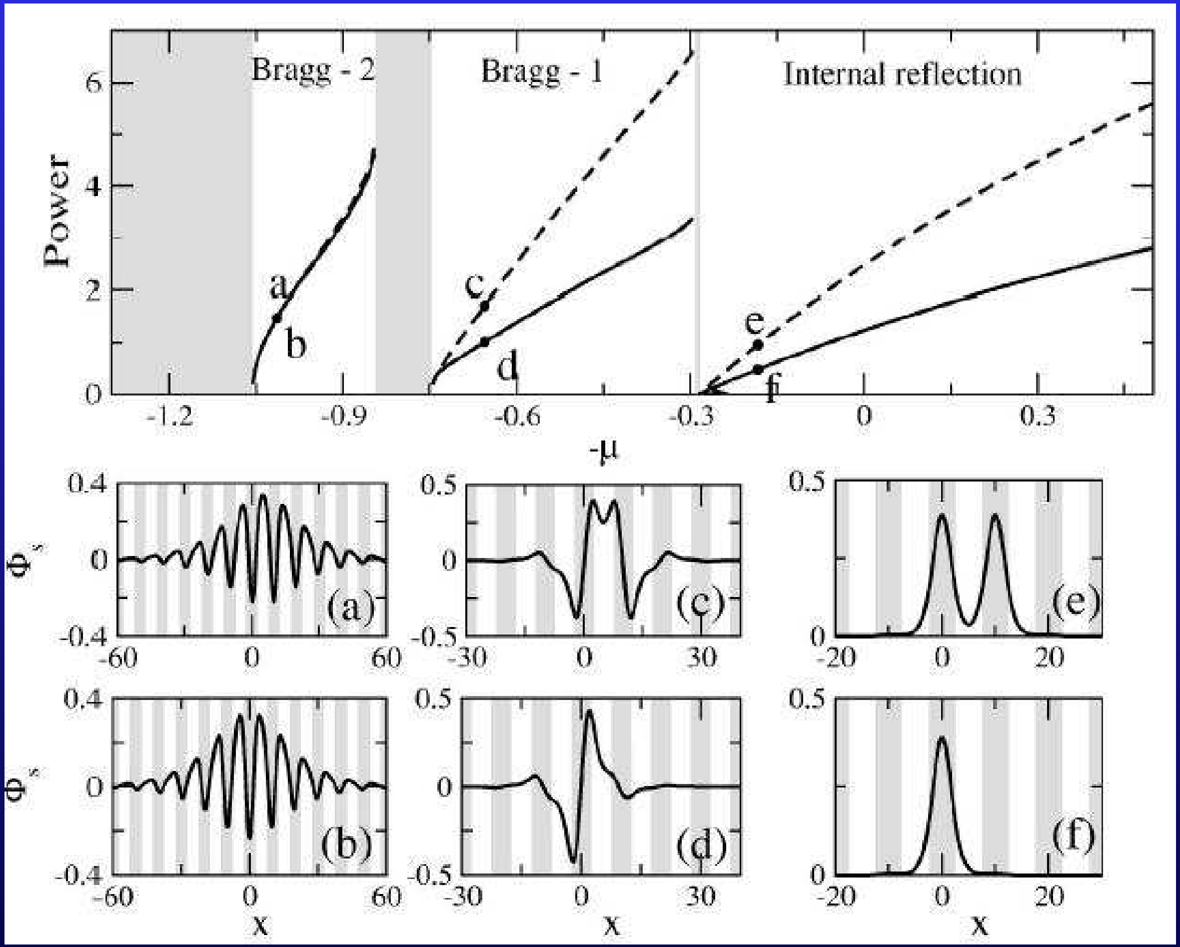
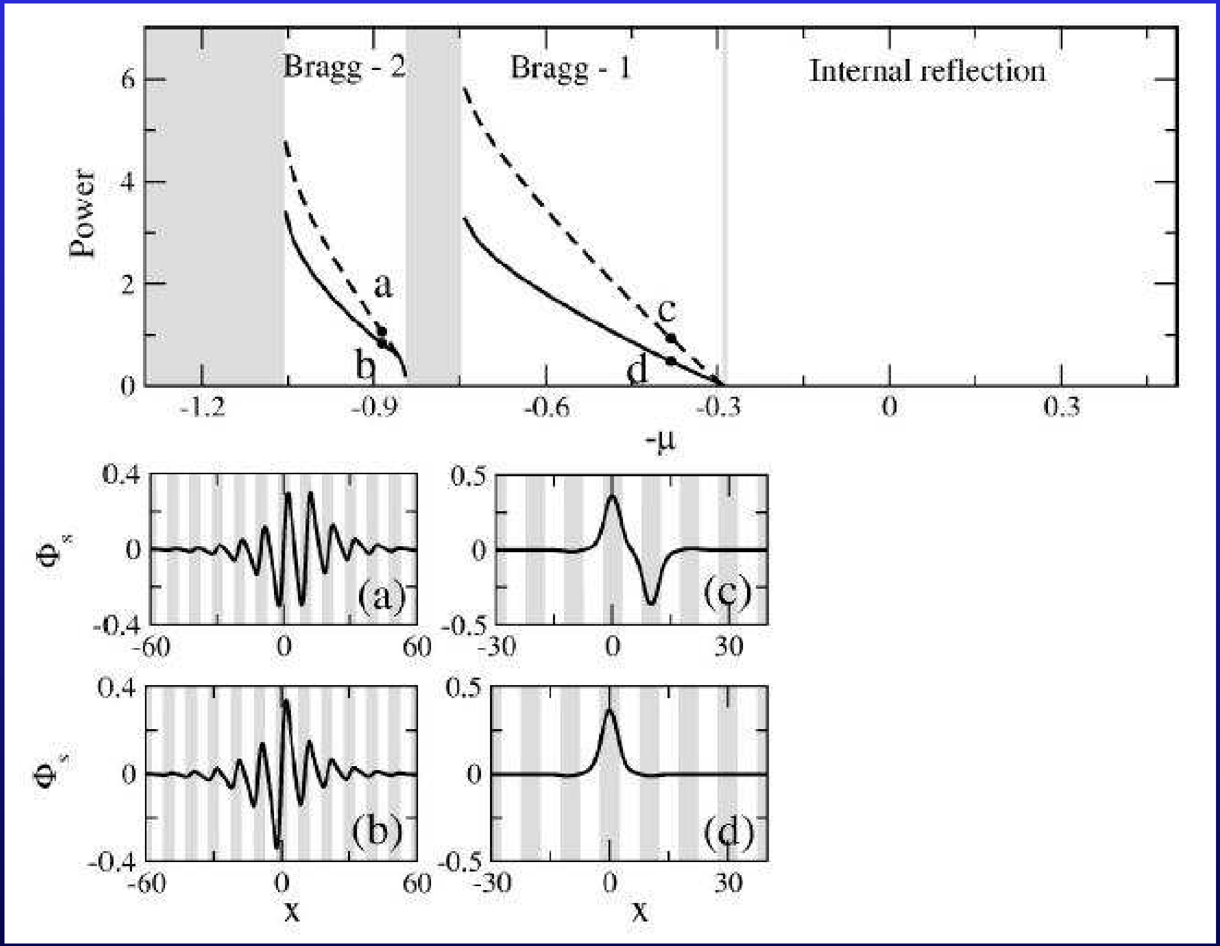


Illustration of solution branches

D.P., A. Sukhorukov, Yu. Kivshar, PRE **70**, 036618 (2004)

$V(x) = V_0 \sin^2(x)$ with $V_0 = 1$ and $\sigma = +1$:



Asymptotic reductions

The nonlinear elliptic problem with a periodic potential can be reduced asymptotically to the following problems:

- Coupled-mode (Dirac) equations for small potentials

$$\begin{cases} ia'(x) + \Omega a + \alpha b = \sigma(|a|^2 + 2|b|^2)a \\ -ib'(x) + \Omega b + \alpha a = \sigma(2|a|^2 + |b|^2)b \end{cases}$$

- Envelope (NLS) equations for finite potentials near band edges

$$a''(x) + \Omega a + \sigma |a|^2 a = 0$$

- Lattice (dNLS) equations for large or long-period potentials

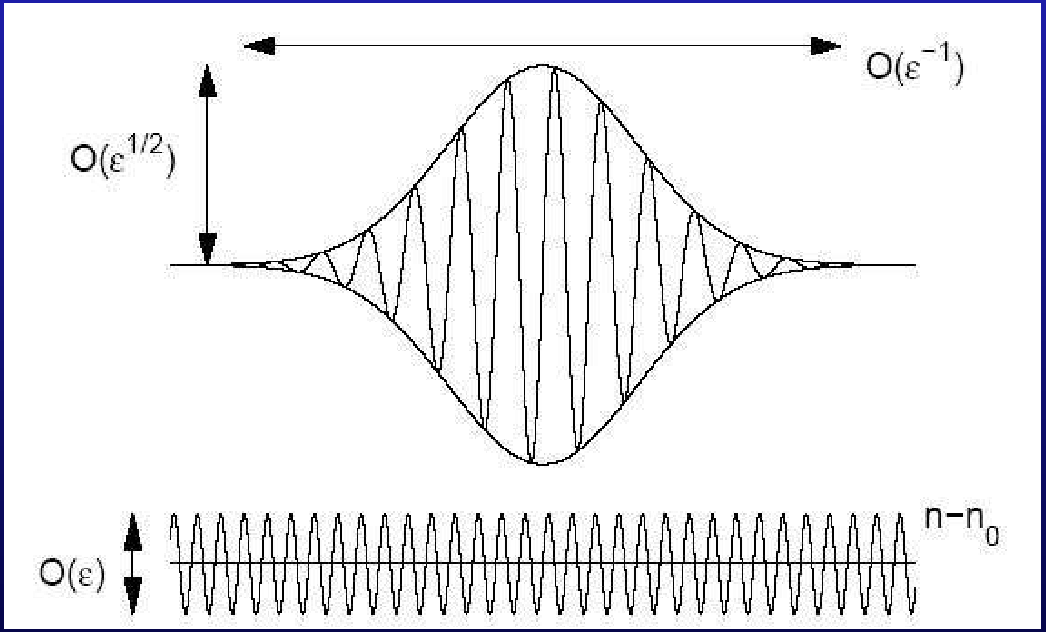
$$\alpha \left(a_{n+1} + a_{n-1}\right) + \Omega a_n + \sigma |a_n|^2 a_n = 0.$$

Localized solutions of reduced equations exist in the analytic form.

Formal coupled-mode theory in 1D

If $V(x) \equiv 0$, then 2π-periodic or 2π-antiperiodic Bloch functions exist for $\omega = \omega_n = \frac{n^2}{4}$, where $n \in \mathbb{Z}$. Let $\omega = \omega_1$ and consider the asymptotic multi-scale expansion

$$E(x,t) = \sqrt{\epsilon}\left[a(\epsilon x, \epsilon t)e^{\frac{ix}{2}} + b(\epsilon x, \epsilon t)e^{-\frac{ix}{2}} + \mathrm{O}(\epsilon)\right]e^{-\frac{it}{4}}.$$



Coupled-mode equations

The vector $(a, b) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}^2$ satisfies asymptotically the coupled-mode system:

$$\begin{cases} i(a_T + a_X) + V_1 b = \sigma(|a|^2 + 2|b|^2)a, \\ i(b_T - b_X) + V_{-1} a = \sigma(2|a|^2 + |b|^2)b, \end{cases}$$

where $X = \epsilon x$, $T = \epsilon t$, and $V_1 = \bar{V}_{-1}$ are Fourier coefficients of $V(x)$ at $e^{\pm ix}$.

The dispersion relation of the linearized coupled-mode equation is

$$(\omega - \omega_1)^2 = \epsilon^2 |V_1|^2 + k^2.$$

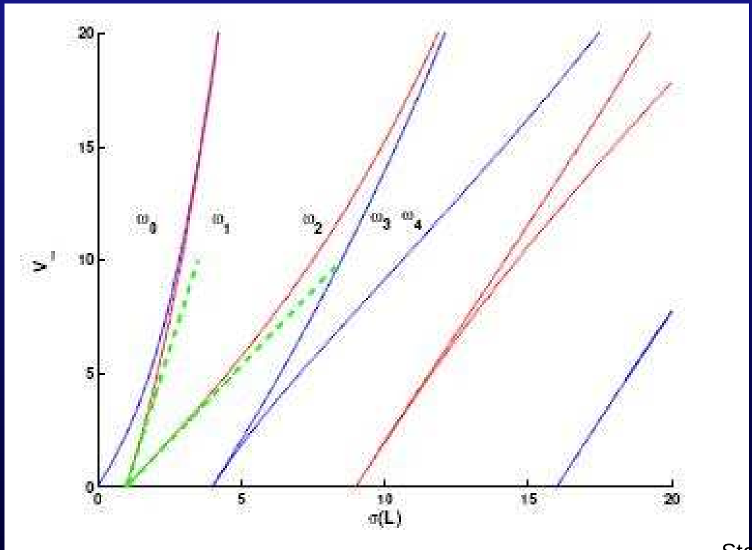
Stationary gap solitons

Stationary gap solitons are obtained in the analytic form

$$a(X,T) = a(X)e^{-i\Omega T}, \quad b(X,T) = b(X)e^{-i\Omega T},$$

where $\kappa = \sqrt{|V_1|^2 - \Omega^2}$ and $|\Omega| < |V_1|$, and

$$a(X) = \bar{b}(X) = \frac{\sqrt{2}}{\sqrt{3}} \frac{\sqrt{|V_1|^2 - \Omega^2}}{\sqrt{|V_1| - \Omega}\cosh(\kappa X) + i\sqrt{|V_1| + \Omega}\sinh(\kappa X)}.$$



Moving gap solitons

Moving gap solitons are obtained in the analytic form

$$a = \left(\frac{1+c}{1-c}\right)^{1/4} A(\xi)e^{-i\mu\tau},\ b = \left(\frac{1-c}{1+c}\right)^{1/4} B(\xi)e^{-i\mu\tau},\ |c| < 1,$$

where

$$\xi = \frac{X - cT}{\sqrt{1-c^2}}, \quad \tau = \frac{T - cX}{\sqrt{1-c^2}}$$

and, since $|A|^2 - |B|^2$ is constant in $\xi \in \mathbb{R}$, then

$$A = \phi(\xi)e^{i\varphi(\xi)}, \qquad B = \bar{\phi}(\xi)e^{i\varphi(\xi)},$$

with ϕ and φ being solutions of the system

$$\varphi' = \frac{-2c\sigma|\phi|^2}{(1-c^2)}, \quad i\phi' = V_1\bar{\phi} - \mu\phi + \sigma\frac{(3-c^2)}{(1-c^2)}|\phi|^2\phi.$$

Questions and Answers

Question 1: Can we justify the use of the coupled-mode theory to approximate stationary gap solitons?

Answer 1: **YES:** we can measure a small approximation error of stationary solutions in $H^1(\mathbb{R})$.

Question 2: Can we justify the use of the coupled-mode theory to approximate moving gap solitons?

Answer 2: **NO:** the small approximation error of traveling solutions is controlled on a large but finite interval and the gap soliton is surrounded by a train of small-amplitude almost-periodic waves.

Time-dependent coupled-mode system

Theorem:[Goodman-Weinstein-Holmes, 2001; Schneider-Uecker, 2001:] Let $(a, b) \in C([0, T_0], H^3(\mathbb{R}, \mathbb{C}^2))$ be solutions of the time-dependent coupled-mode system for a fixed $T_0 > 0$. There exists $\epsilon_0, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the Gross–Pitaevskii equation has a local solution $E(x, t)$ and

$$\|E(x,t) - \sqrt{\epsilon}\left[a(\epsilon x, \epsilon t)e^{i(kx-\omega t)} + b(\epsilon x, \epsilon t)e^{i(-kx-\omega t)}\right]\|_{H^1(\mathbb{R})} \leq C\epsilon$$

for some (k, ω) and any $t \in [0, T_0/\epsilon]$.

Remark: We would like to consider stationary and moving gap solitons in $H^1(\mathbb{R})$ for all $t \in \mathbb{R}$.

Main theorem for stationary solutions

Assumption: Let $V(x)$ be a smooth 2π-periodic real-valued function with zero mean and symmetry $V(x) = V(-x)$ on $x \in \mathbb{R}$, such that

$$V(x) = \sum_{m \in \mathbb{Z}} V_m e^{imx} : \quad \sum_{m \in \mathbb{Z}} (1 + m^2)^s |V_m|^2 < \infty,$$

for some $s \geq 0$, where $V_0 = 0$ and $V_m = V_{-m} = \bar{V}_{-m}$.

Definition: The gap soliton of the coupled-mode system is said to be a reversible homoclinic orbit if (a, b) decays to zero as $|X| \to \infty$ and $a(X) = \bar{a}(-X), b(X) = \bar{b}(-X)$.

Remark: If $V(x) = V(-x)$ and $U(x)$ is a solution of $\nabla^2 U + \omega U = V(x) U + \sigma |U|^2 U$, then $\bar{U}(-x)$ is also a solution.

Spaces for the main theorem

Let $U(x)$ be represented by the Fourier transform

$$U(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{U}(k) e^{ikx} dk, \qquad \hat{U}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} U(x) e^{-ikx} dx,$$

in the vector space

$$\hat{U} \in L^1_q(\mathbb{R}) : \quad \|\hat{U}\|_{L^1_q(\mathbb{R})} = \int_{\mathbb{R}} (1+k^2)^{q/2} |\hat{U}(k)| dk < \infty.$$

Properties:

1) If $\hat{U} \in L^1_q(\mathbb{R})$, then $U(x)$ is n-times continuously differentiable on $x \in \mathbb{R}$ for $0 \leq n \leq [q]$.

2) If $\hat{U} \in L^1_q(\mathbb{R})$, then $U \in H^q(\mathbb{R})$.

3) $L^1_q(\mathbb{R})$ is a Wiener algebra $\left\| \hat{U} \star \hat{W} \right\|_{L^1_q} \leq \|\hat{U}\|_{L^1_q} \|\hat{W}\|_{L^1_q}$.

Main Theorem

Theorem: Let $V(x)$ satisfy the assumption and $V_n \neq 0$ for a fixed $n \in \mathbb{N}$. Let $\omega = \frac{n^2}{4} + \epsilon\Omega$ with $|\Omega| < |V_n|$. Let (a, b) be a reversible homoclinic orbit of the coupled-mode system. Then, there exists $\epsilon_0, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the nonlinear elliptic problem has a non-trivial solution $U(x)$ and

$$\|U(x) - \sqrt{\epsilon}\left[a(\epsilon x)e^{\frac{inx}{2}} + b(\epsilon x)e^{-\frac{inx}{2}}\right]\|_{H^q(\mathbb{R})} \leq C\epsilon^{5/6},$$

for any $q \geq 0$. Moreover, the solution $U(x)$ is real-valued, continuous on $x \in \mathbb{R}$, and $\lim_{|x| \to \infty} U(x) = 0$.

Remarks: 1) We do not prove that $U(x)$ decays exponentially at infinity. 2) The power of $\epsilon^{5/6}$ can be extended to any ϵ^p for $\frac{1}{2} < p < 1$.

Steps of the proof

1. Convert the problem

$$U''(x) + \omega U(x) = \epsilon V(x)U(x) + \epsilon\sigma |U(x)|^2 U(x),$$

to the integral equation

$$\left(\omega - k^2\right) \hat{U}(k) = \epsilon \sum_{m\in\mathbb{Z}} V_m \hat{U}(k-m)$$
$$+\epsilon\sigma \int\int \hat{U}(k_1)\hat{\bar{U}}(k_2)\hat{U}(k-k_1+k_2)dk_1 dk_2$$

2. If $\mathbf{V} \in l^2_{s+q}(\mathbb{Z})$ for any $s > \frac{1}{2}$ and $q \geq 0$, then the vector field of the right-hand-side of the integral equation maps an element of $L^1_q(\mathbb{R})$ to an element of $L^1_q(\mathbb{R})$.

Steps of the proof

3. Decompose the solution $\hat{U}(k)$ into three parts

$$\hat{U}(k) = \hat{U}_+(k)\chi_{\mathbb{R}'_+}(k) + \hat{U}_-(k)\chi_{\mathbb{R}'_-}(k) + \hat{U}_0(k)\chi_{\mathbb{R}'_0}(k)$$

with a compact support on

$$\mathbb{R}'_\pm = \left[\pm n/2 - \epsilon^{2/3}, \pm n/2 + \epsilon^{2/3}\right], \quad \mathbb{R}'_0 = \mathbb{R}\backslash(\mathbb{R}'_+ \cup \mathbb{R}'_-),$$

where $\inf_{k\in\mathbb{R}'_0} |n^2/4 - k^2| \geq C\epsilon^{2/3}$.

4. There exists a unique map $\hat{U}_\epsilon : L^1_q(\mathbb{R}'_+) \times L^1_q(\mathbb{R}'_-) \mapsto L^1_q(\mathbb{R}'_0)$ such that $\hat{U}_0(k) = \hat{U}_\epsilon(\hat{U}_+, \hat{U}_-)$ and

$$\forall|\epsilon| < \epsilon_0 : \quad \|\hat{U}_0(k)\|_{L^1_q(\mathbb{R}'_0)} \leq \epsilon^{1/3} C \left(\|\hat{U}_+\|_{L^1_q(\mathbb{R}'_+)} + \|\hat{U}_-\|_{L^1_q(\mathbb{R}'_-)}\right).$$

Steps of the proof

5. Write projections to the new amplitudes for the singular part

$$\hat{U}_+(k) = \frac{1}{\epsilon}\hat{A}\left(\frac{k - n/2}{\epsilon}\right), \quad \hat{U}_-(k) = \frac{1}{\epsilon}\hat{B}\left(\frac{k + n/2}{\epsilon}\right),$$

where $\hat{A}(p)$, $\hat{B}(p)$ are defined on $p \in \mathbb{R}_0 = [-\epsilon^{-1/3}, \epsilon^{-1/3}]$ and

$$\|\hat{U}_+\|_{L^1_q(\mathbb{R}'_+)} \le C\|\hat{A}\|_{L^1_q(\mathbb{R}_0)}, \quad \|\hat{U}_-\|_{L^1_q(\mathbb{R}'_-)} \le C\|\hat{B}\|_{L^1_q(\mathbb{R}_0)}.$$

6. Prove persistence of gap soliton solutions in the coupled-mode system on $p \in \mathbb{R}_0$, e.g.

$$(\Omega - np)\,\hat{A}(p) + V_n\hat{B}(p) - \sigma\text{Conv.Int.}$$
$$= \epsilon p^2\hat{A}(p) + \epsilon^{1/3}\hat{R}_a(\hat{A}, \hat{B}, \hat{U}_\epsilon(\hat{A}, \hat{B})).$$

Steps of the proof

7. Analyze the reminder terms, e.g.

$$\|\hat{R}_a\|_{L^1_q(\mathbb{R}_0)} \leq C_a \|\hat{A}\|_{L^1_q(\mathbb{R}_0)}, \quad \epsilon \|p^2 \hat{A}(p)\|_{L^1_q(\mathbb{R}_0)} \leq \epsilon^{1/3} \|\hat{A}(p)\|_{L^1_q(\mathbb{R}_0)},$$

8. Solve the system $\hat{\mathbf{N}}(\hat{\mathbf{A}}) = \hat{\mathbf{R}}(\hat{\mathbf{A}})$ for $\hat{A} = \hat{a} + \hat{\tilde{A}}$ by fixed-point iterations

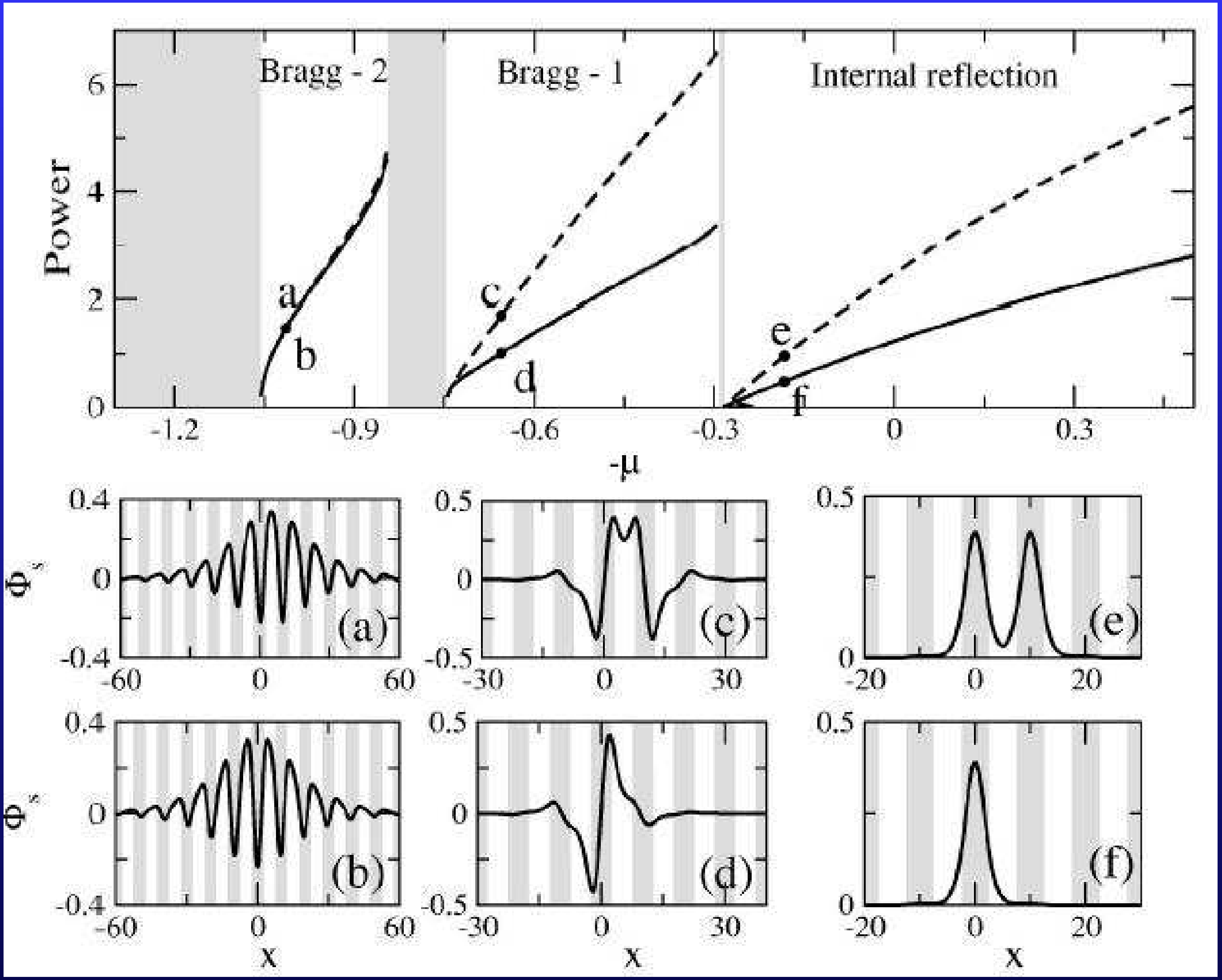
$$\hat{L}\hat{\tilde{\mathbf{A}}} = \hat{\mathbf{R}}(\hat{\mathbf{a}} + \hat{\tilde{\mathbf{A}}}) - \left[\hat{\mathbf{N}}(\hat{\mathbf{a}} + \hat{\tilde{\mathbf{A}}}) - \hat{\mathbf{L}}\hat{\tilde{\mathbf{A}}}\right], \quad \hat{\mathbf{L}} = \mathbf{D}_{\hat{\mathbf{a}}}\hat{\mathbf{N}}(\hat{\mathbf{a}}),$$

where $\hat{L}$ is a linearized operator for the coupled-mode system.

9. Analyze the truncation terms, e.g.

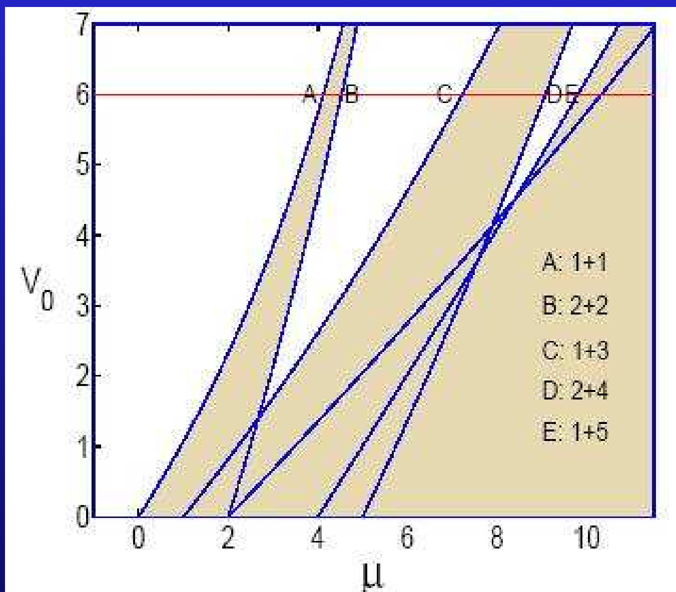
$$\|\hat{A} - \hat{a}\|_{L^1_{q+1}(\mathbb{R}\backslash\mathbb{R}_0)} \leq \|\hat{A} - \hat{a}\|_{L^1_{q+1}(\mathbb{R})} \leq \epsilon^{1/3} C \|\hat{R}_a\|_{L^1_q(\mathbb{R})}.$$

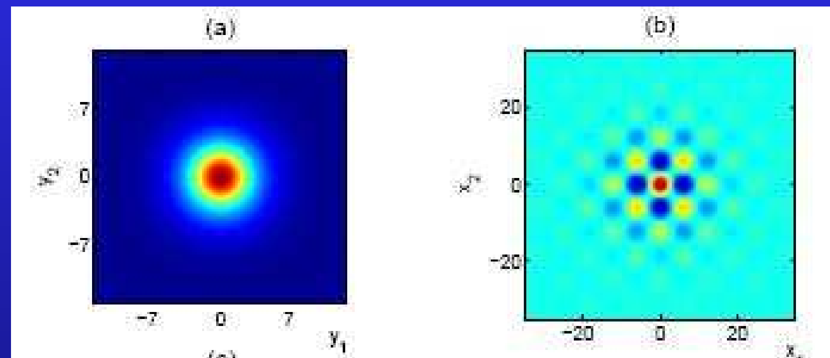
Intermission

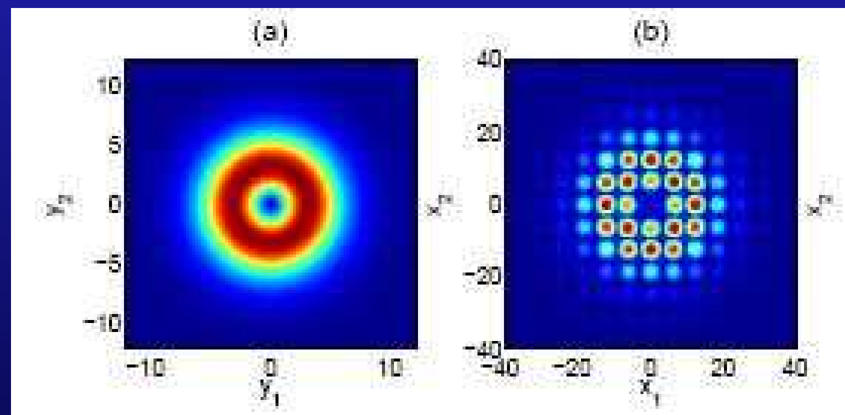


Intermission

T. Dohnal, D.P., G. Schneider, submitted to J. Nonlinear Science (2007) - $V(x_1, x_2) = V(x_1) + V(x_2)$.







Spatial dynamics formulation

Set $E(x,t) = e^{-i\omega t}\psi(x,y)$ with $y = x - ct$ and a parameter ω. For traveling solutions, $c \neq 0$ and we set $c > 0$. Then,

$$\left(\omega - ic\partial_y + \partial_x^2 + 2\partial_x\partial_y + \partial_y^2\right)\psi = \epsilon V(x)\psi + \epsilon\sigma|\psi|^2\psi.$$

We consider functions $\psi(x,y)$ being 2π-periodic or 2π-antiperiodic in x and bounded in y. Therefore,

$$\psi(x,y) = \sum_{m \in \mathbb{Z}'} \psi_m(y) e^{\frac{i}{2}mx},$$

such that $\psi_m(y)$ satisfy the nonlinear system of coupled ODEs:

$$\psi_m'' + i(m-c)\psi_m' + \left(\omega - \frac{m^2}{4}\right)\psi_m = \epsilon \sum_{m_1 \in \mathbb{Z}'} V_{m-m_1}\psi_{m_1} + \epsilon \mathrm{N.T.}$$

Eigenvalues of the spatial dynamics

Linearization of the system with $\psi_m(y) = e^{\kappa y}\delta_{m,m_0}$ gives roots $\kappa = \kappa_m$ in the quadratic equation

$$\kappa^2 + i(m-c)\kappa + \omega - \frac{m^2}{4} = 0, \qquad \forall m \in \mathbb{Z}'.$$

- If $\omega = \frac{n^2}{4}$, there is a double zero root $\kappa = 0$.
- For $m > m_0 = \left[\frac{n^2+c^2}{2c}\right]$, all roots κ are complex-valued.
- For $m \leq m_0$, all roots κ are purely imaginary and semi-simple of maximal multiplicity three.

M. Groves, G. Schneider, Comm. Math. Phys. **219**, 489 (2001)

Main theorem for traveling solutions

Theorem: There exists $\epsilon_0, L, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the Gross–Pitaevskii equation has a solution in the form $E(x,t) = e^{-i\omega t}\psi(x,y)$, where $y = x - ct$ and the function $\psi(x,y)$ is a periodic (anti-periodic) function of x for even (odd) n, satisfying the reversibility constraint $\psi(x,y) = \bar{\psi}(x,-y)$, and

$$\left|\psi(x,y) - \epsilon^{1/2}\left(a_\epsilon(\epsilon y)e^{\frac{inx}{2}} + b_\epsilon(\epsilon y)e^{-\frac{inx}{2}}\right)\right| \leq C_0\epsilon^{N+1/2},$$

for all $x \in \mathbb{R}$ and $y \in [-L/\epsilon^{N+1}, L/\epsilon^{N+1}]$. Here $a_\epsilon(Y) = a(Y) + \mathrm{O}(\epsilon)$ on $Y = \epsilon y \in \mathbb{R}$ is an exponentially decaying reversible solution, while $a(Y)$ is a solution of the coupled-mode system with $Y = X - cT$.

Summary

- We have justified approximations of gap solitons by the coupled-mode equations for small one-dimensional potentials.
- Similar methods (with the use of the Fourier–Bloch transform) are developed to justify the continuous and discrete NLS equations for finite and large multi-dimensional separable periodic potentials.
- Moving gap solitons do not generally exist because of an infinite set of purely imaginary eigenvalues in the spatial dynamics formulations of the problem.