Spectrum of the linearized NLS problem

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Stability problem in Hamiltonian systems

- Hamiltonian PDE
$$\frac{du}{dt} = J\nabla h(u), \quad u(t) \in X(\mathbb{R}^n, \mathbb{R}^m)$$
where $J^+ = -J$ and $h : X \mapsto \mathbb{R}$

- Linearization at the stationary solution
$$u(t) = u_0 + ve^{\lambda t},$$
where $u_0 \in X(\mathbb{R}^n, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$

- Spectral problem
$$JHv = \lambda v,$$
where $H^+ = H$ and $v \in X(\mathbb{R}^n, \mathbb{C}^m)$

Main questions

- Let stationary solutions u_0 decay exponentially as $|x| \to \infty$
- Let operator J be invertible
- Let operator H have positive continuous spectrum
- Let operator H have finitely many isolated eigenvalues
- Let operator JH have continuous spectrum at the imaginary axis

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Is there a relation between isolated and embedded eigenvalues of JH and isolated eigenvalues of H?

Is there a relation between unstable eigenvalues of JH with $\mathrm{Re}(\lambda) > 0$ and negative eigenvalues of H?

Classes of Hamiltonian evolution equations

- Nonlinear Schrödinger equation (NLS)
$$i\psi_t = -\psi_{xx} + U(x)\psi + F(|\psi|^2)\psi$$
(J, H) satisfy the main assumptions

- Korteweg–De Vries equation (KdV)
$$u_t + \partial_x \left(f(u) + u_{xx}\right) = 0$$
J is not invertible but H satisfy the main assumptions

- Massive Thirring model (MTM)
$$i(u_t + u_x) + v + \partial_{\bar{u}} W(u, \bar{u}, v, \bar{v}) = 0,$$
$$i(v_t - v_x) + u + \partial_{\bar{v}} W(u, \bar{u}, v, \bar{v}) = 0$$
J is invertible but H have positive and negative continuous spectrum

Review of other results

Grillakis, Shatah, Strauss, 1990

- If H has no negative eigenvalue, then JH has no unstable eigenvalues.
- If H has odd number of negative eigenvalues, then JH has at least one real unstable eigenvalue.
- Number of unstable eigenvalues of JH is bounded by the number of negative eigenvalues of H.

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Kapitula, Kevrekidis, Sandstede, 2004

- Closure relation for negative index

$$N_{\text{unstable}}(JH) + N_{\text{negative Krein}}(JH) = N_{\text{negative}}(H)$$

Review of our results: formalism

$$i\psi_t = -\Delta\psi + U(x)\psi + F(|\psi|^2)\psi$$

- Assume that there exist exponentially decaying C^∞ solutions
$$-\Delta\phi + U(x)\phi + F(\phi^2)\phi + \omega\phi = 0,$$
where $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ and $\omega > 0$.

- Assume that $U(x)$ decay exponentially and $F(u) \in C^\infty$, $F(0) = 0$

- Apply the linearization transformation,
$$\psi(x,t) = e^{i\omega t}\left(\phi(x) + \varphi(x)e^{-izt} + \bar{\theta}(x)e^{i\bar{z}t}\right),$$
where $(\varphi, \theta) : \mathbb{R}^n \mapsto \mathbb{C}^2$ and $z = i\lambda \in \mathbb{C}$.

Review of our results : formalism

- The eigenvalue problem becomes

$$\sigma_3 H \boldsymbol{\psi} = z\boldsymbol{\psi},$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} -\Delta + \omega + f(x) & g(x) \\ g(x) & -\Delta + \omega + f(x) \end{pmatrix},$$

and

$$f(x) = U(x) + F(\phi^2) + \phi^2 F'(\phi^2), \quad g(x) = \phi^2 F'(\phi^2).$$

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$$f(x) = U(x) + F(\phi^2) + \phi^2 F'(\phi^2), \quad g(x) = \phi^2 F'(\phi^2).$$

- Equivalent form:

$$L_+ u = zw, \qquad L_- w = zu,$$

where

$$L_\pm = -\Delta + \omega + f(x) \pm g(x)$$

and

$$\boldsymbol{\psi} = (u + w, u - w)^T.$$

Review of our main results : formalism

- Linearized energy

$$h = \frac{1}{2}\langle \boldsymbol{\psi}, H\boldsymbol{\psi}\rangle = (u, L_+ u) + (w, L_- w)$$

- Skew-orthogonal projections

$$\langle \boldsymbol{\psi}^*, \boldsymbol{\psi}\rangle = \frac{1}{2}\langle \sigma_3\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = (u, w) + (w, u)$$

- Constrained subspace: no kernel of JH

$$X_0 = \{\boldsymbol{\psi} \in L^2 : \langle \sigma_3\boldsymbol{\psi}_0, \boldsymbol{\psi}\rangle = \langle \sigma_3\boldsymbol{\psi}_1, \boldsymbol{\psi}\rangle = 0\},$$

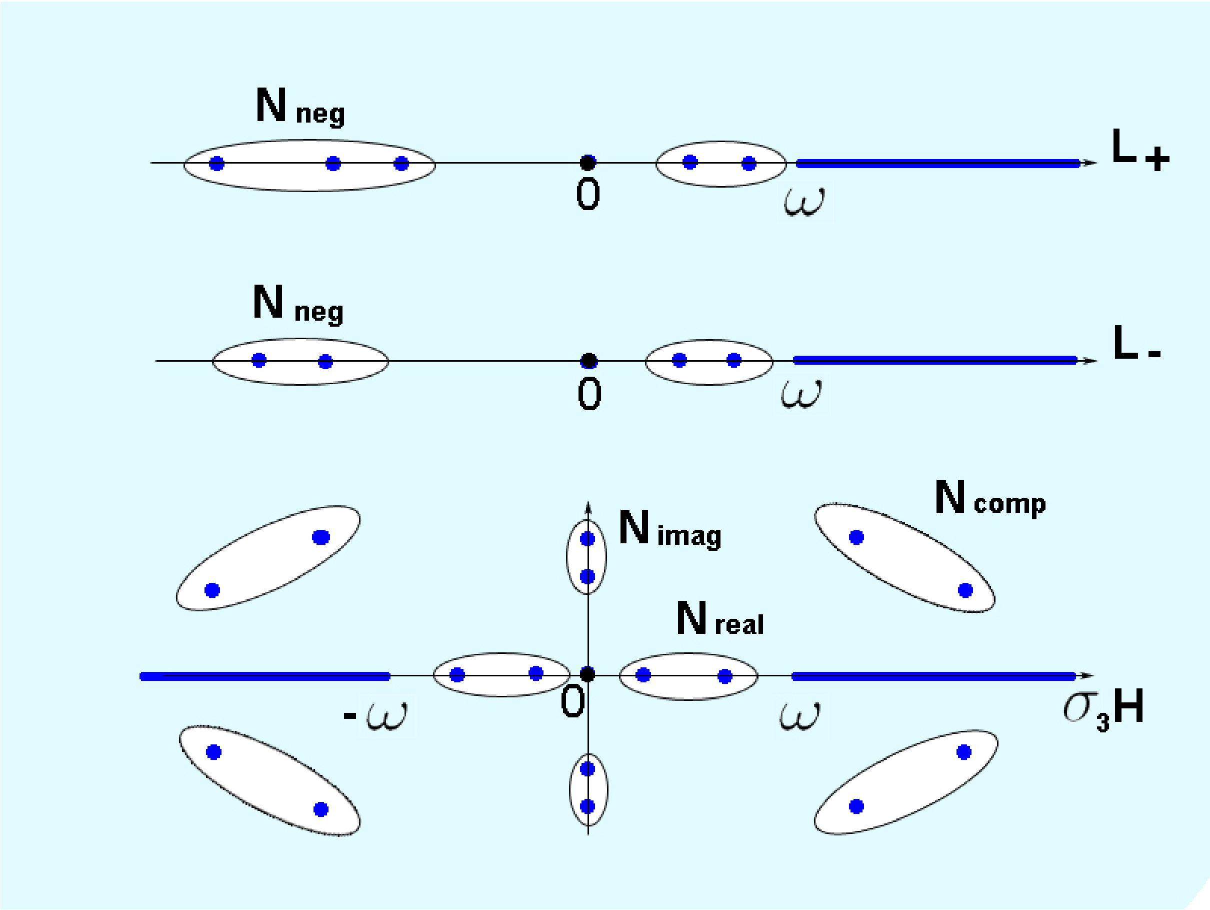
  where

$$H\boldsymbol{\psi}_0 = 0, \qquad \sigma_3 H\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0$$

- Constrained subspace: no point spectrum of JH

$$X_c = \{\boldsymbol{\psi} \in X_0 : \{\langle \sigma_3\boldsymbol{\psi}_j, \boldsymbol{\psi}\rangle = 0\}_{z_j \in \sigma_p(JH)}\},$$

Spectrum of (L_+, L_-) and JH



Review of our main results

- Assuming that $\dim(\ker(H)) = 1$,

$$N_{\text{neg}}(H)\Big|_{X_0} = N_{\text{neg}}(H)\Big|_{L^2} - p(\omega),$$

where $p(\omega) = 1$ for $\partial_\omega \|\phi\|_{L^2}^2 > 0$ and $p(\omega) = 0$ otherwise.

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- Assuming that all eigenvalues of JH are semi-simple,

$$N_{\text{neg}}(H)\Big|_{X_c} = N_{\text{neg}}(H)\Big|_{X_0} - 2N^-_{\text{real}}(JH) - N_{\text{imag}}(JH) - 2N_{\text{comp}}(JH)$$

where $N^-_{\text{neg}}(JH)$ correspond to negative Krein signatures

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- Assuming end-point conditions and simple embedded eigenvalues,

$$\forall \boldsymbol{\psi} \in X_c : \quad \langle \boldsymbol{\psi}, H\boldsymbol{\psi} \rangle > 0$$

Methods of proofs

I. Bounds on the number of isolated eigenvalues:

- Calculus of constrained Hilbert spaces
- Sylvester's Inertia Law
- Rayleigh-Ritz Theorem
- Pontryagin–Krein spaces with sign-indefinite metric

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II. Positivity of the essential spectrum:

- Kato's wave operator formalism
- Wave function decomposition
- Smoothing decay estimate on the linearized time evolution

Quadratic forms and skew-symmetric orthogonality

$$L_+ u = z w, \qquad L_- w = z u,$$

- Let $z = z_0$ be a simple real eigenvalue with the eigenvector (u_R, w_R)

$$(u_R, L_+ u_R) = (w_R, L_- w_R) \neq 0$$

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- Let $z = i z_0$ be a simple imaginary eigenvalue with the eigenvector $(u_R, i w_I)$

$$(u_R, L_+ u_R) = -(w_I, L_- w_I) \neq 0$$

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- Let $z = z_0$ be a simple complex eigenvalue with the eigenvector $(u_R + i u_I, w_R + i w_I)$

$$(u, L_+ u) = (w, L_- w) = (\mathbf{c}, M \mathbf{c}) \neq 0,$$

where

$$M = \begin{pmatrix} (u_R, L_+ u_R) & (u_R, L_+ u_I) \\ (u_I, L_+ u_R) & (u_I, L_+ u_I) \end{pmatrix} = \begin{pmatrix} (w_R, L_- w_R) & (w_R, L_- w_I) \\ (w_I, L_- w_R) & (w_I, L_- w_I) \end{pmatrix}$$

Constrained Hilbert space

- Let z_i and z_j be two distinct eigenvalues with eigenvectors (u_i, w_i) and (u_j, w_j)

$$(u_i, w_j) = (w_i, u_j) = 0 \quad \forall i \neq j$$

- Constrained subspace with no point spectrum of JH

$$X_c = \{(u, w) \in L^2 : \ \{(u, w_j) = 0, (w, u_j) = 0\}_{z_j \in \sigma_p(JH)}\},$$

- Main question:

$$N_{\rm neg}(H)\Big|_{L^2} - N_{\rm neg}(H)\Big|_{X_c} = ?$$

Main Theorem I : bounds on isolated eigenvalues

- Let L be a self-adjoint operator on $X \subset L^2$ with a finite negative index N_{neg}, empty kernel, and positive essential spectrum.
- Let X_c be the constrained linear subspace on linearly independent vectors:

$$X_c = \left\{ v \in X : \ \{(v, v_j) = 0\}_{j=1}^{N} \right\}$$

- Let the matrix A be defined by

$$A_{i,j} = (v_i, L^{-1} v_j), \qquad 1 \leq i, j \leq N$$

- Then,

$$N_{\text{neg}}(L)\Big|_{X_c} = N_{\text{neg}}(L)\Big|_{X} - N_{\text{neg}}(A), \qquad 1 \leq i, j \leq N$$

Application of the Main Theorem I

- Consider two matrices A_+ and A_- for two operators L_+ and L_- constrained with two sets of eigenfunctions $\{w_j\}$ and $\{u_j\}$

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- Due to skew-orthogonality, the matrices $A_\pm$ are block-diagonal.
- For real eigenvalue $z = z_0$ with the eigenvector (u_R, w_R)

$$A^+_{j,j} = A^-_{j,j} = \frac{1}{z_0^2}(u_R, L_+ u_R) = \frac{1}{z_0^2}(w_R, L_- w_R).$$

- For imaginary eigenvalue $z = iz_0$ with the eigenvector (u_R, iw_I)

$$A^+_{j,j} = -A^-_{j,j} = \frac{1}{z_0^2}(u_R, L_+ u_R) = -\frac{1}{z_0^2}(w_I, L_- w_I).$$

Application of the Main Theorem I

- For complex eigenvalue $z = z_R + iz_I$ with the eigenvector $(u_R + iu_I, w_R + iw_I)$

$$A^{+}_{i,j} = A^{-}_{i,j} = Z^2 M, \quad Z = \frac{1}{z_R^2 + z_I^2} \begin{pmatrix} z_R & z_I \\ -z_I & z_R \end{pmatrix}.$$

Application of the Main Theorem I

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- Individual counts of eigenvalues

$$N_{\text{neg}}(L_+)\Big|_{X_c} = N_{\text{neg}}(L_+)\Big|_{X_0} - N^{-}_{\text{real}}(JH) - N^{-}_{\text{imag}}(JH) - N_{\text{comp}}(JH)$$

and

$$N_{\text{neg}}(L_-)\Big|_{X_c} = N_{\text{neg}}(L_-)\Big|_{X_0} - N^{-}_{\text{real}}(JH) - N^{+}_{\text{imag}}(JH) - N_{\text{comp}}(JH)$$

- End of proof of I

Projections and decompositions

- Birman–Schwinger representation of the potentials
$$V(x) = \begin{pmatrix} f(x) & g(x) \\ g(x) & f(x) \end{pmatrix} = B^*(x)A(x)$$
and of the spectral problem
$$(\sigma_3(-\Delta + \omega) - z)\,\boldsymbol{\psi} = -B^* A \boldsymbol{\psi},$$
such that
$$(I + Q_0(z))\,\boldsymbol{\Psi} = \mathbf{0}, \qquad Q_0(z) = A\left(\sigma_3(-\Delta+\omega) - z\right)^{-1} B^*,$$

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- The set of isolated eigenvalues of $\sigma_3 H$ is finite
- No resonances occur in the interior points
- There exists a spectrum-invariant Jordan-block decomposition
$$L^2 = \sum_{z\in\sigma_p(JH)} N_g(JH-z) \oplus X_c$$

Main Theorem II : positivity of the essential spectrum

- Assume that no resonance exist at the endpoints of $\sigma_e(JH)$
- Assume that no multiple embedded eigenvalues exist in $\sigma_e(JH)$
- $\mathbb{R}^n = \mathbb{R}^3$

Main Theorem II : positivity of the essential spectrum

- Assume that no resonance exist at the endpoints of $\sigma_e(JH)$
- Assume that no multiple embedded eigenvalues exist in $\sigma_e(JH)$
- $\mathbb{R}^n = \mathbb{R}^3$
- There exists isomorphisms between Hilbert spaces

$$W : L^2 \mapsto X_c, \qquad Z : X_c \mapsto L^2$$

- W and Z are inverse of each other, where

$$\forall\, u \in X_c, \forall\, v \in L^2 : \langle Zu, v\rangle = \langle u, v\rangle$$

$$+ \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle A(\sigma_3 H - \lambda - i\epsilon)^{-1} u, B(\sigma_3(-\Delta + \omega) - \lambda - i\epsilon)^{-1} v\rangle d\lambda,$$

Application of the Main Theorem II

- It follows from the wave operators that

$$W^*\sigma_3 = \sigma_3 Z, \quad Z^*\sigma_3 = \sigma_3 W, \quad Z\sigma_3 H = \sigma_3(-\Delta + \omega)Z.$$

Application of the Main Theorem II

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$$W^*\sigma_3 = \sigma_3 Z, \quad Z^*\sigma_3 = \sigma_3 W, \quad Z\sigma_3 H = \sigma_3(-\Delta + \omega)Z.$$

- If $\boldsymbol{\psi} \in X_c$, there exists $\hat{\boldsymbol{\psi}} \in L^2$, such that $\boldsymbol{\psi} = W\hat{\boldsymbol{\psi}}$.

- Then

$$\begin{aligned}\langle \boldsymbol{\psi}, H\boldsymbol{\psi}\rangle &= \langle W\hat{\boldsymbol{\psi}}, \sigma_3\sigma_3 H W\hat{\boldsymbol{\psi}}\rangle = \langle W\hat{\boldsymbol{\psi}}, (\sigma_3 H)^* Z^*\sigma_3\hat{\boldsymbol{\psi}}\rangle \\ &= \langle \sigma_3(-\Delta+\omega)ZW\hat{\boldsymbol{\psi}}, \sigma_3\hat{\boldsymbol{\psi}}\rangle = \langle(-\Delta+\omega)I\hat{\boldsymbol{\psi}}, \hat{\boldsymbol{\psi}}\rangle > 0.\end{aligned}$$

- End of proof of II.

Extensions of the main results

- Fermi Golden Rule for an embedded real eigenvalue
  - It disappears if it has positive energy
  - It becomes complex eigenvalue if it has negative energy

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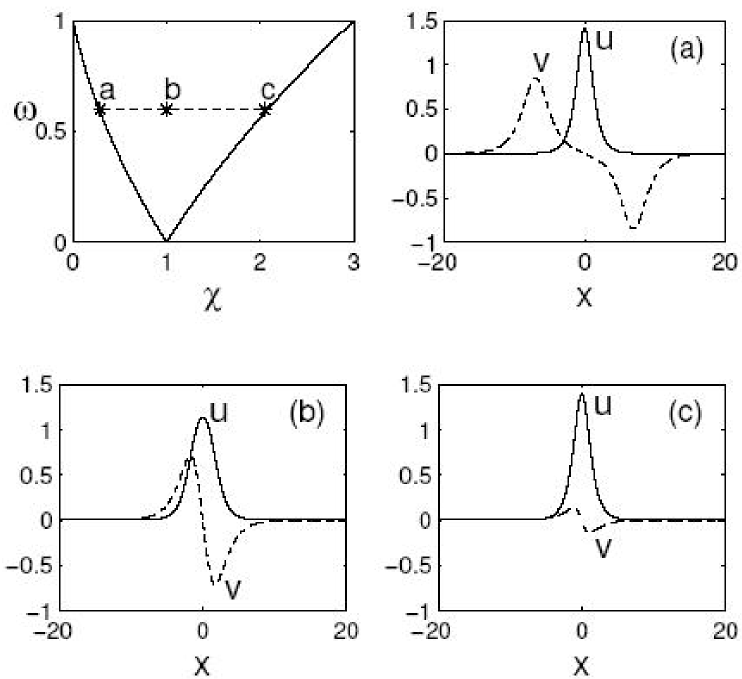
- Weighted spaces for endpoint resonance and eigenvalue
  - Resonance results in real eigenvalue of positive energy
  - Eigenvalue repeats the scenario of embedded eigenvalue

Example: two coupled NLS equations

$$i\psi_{1t} + \psi_{1xx} + \left(|\psi_1|^2 + \chi|\psi_2|^2\right)\psi_1 = 0$$
$$i\psi_{2t} + \psi_{2xx} + \left(\chi|\psi_1|^2 + |\psi_2|^2\right)\psi_2 = 0$$



Example: two coupled NLS equations

- Lyapunov-Schmidt reductions near local bifurcation boundary

$$\psi_1 = e^{it} \left(\sqrt{2}\, \mathrm{sech} x + O(\epsilon^2) \right), \quad \psi_2 = e^{i\omega t} \left(\epsilon \phi_n(x) + O(\epsilon^3) \right),$$

where

$$\left(-\partial_x^2 + \omega_n(\chi) - 2\chi\, \mathrm{sech}^2(x) \right) \phi_n(x) = 0$$

- By continuity of eigenvalues, we count isolated eigenvalues

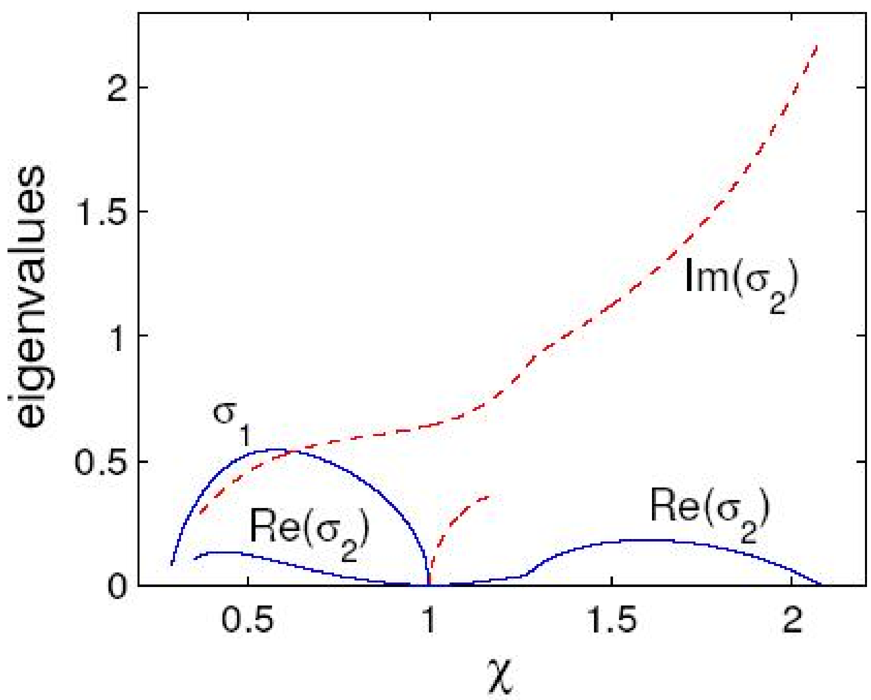
$$N_{\mathrm{neg}}(H) = 2n,$$

where n is the number of zeros of $\phi_n(x)$. Therefore,

$$2N_{\mathrm{real}}^{-}(JH) + N_{\mathrm{imag}}(JH) + 2N_{\mathrm{comp}}(JH) = 2n$$

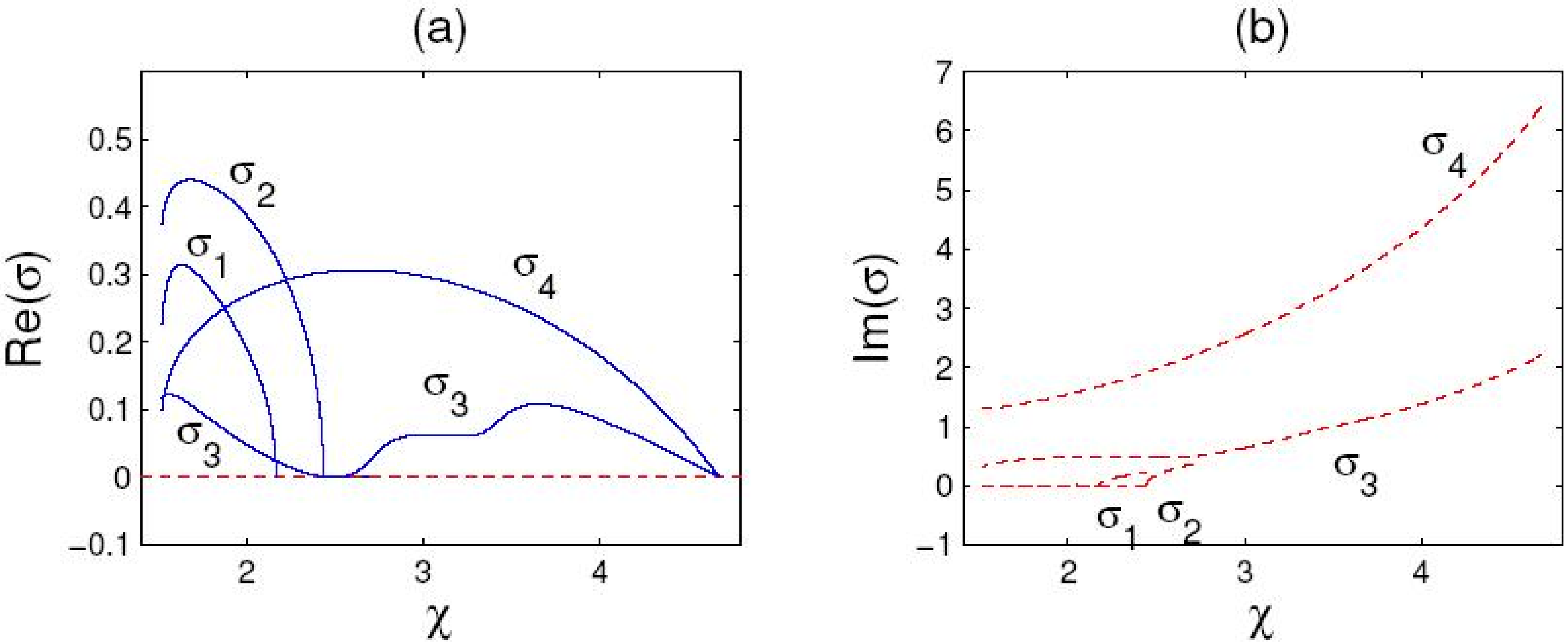
Example: two coupled NLS equations

$n = 1$



Example: two coupled NLS equations

$n = 2$



Work in progress

- Bounds on $N_{\mathrm{real}}^{+}(JH)$ in terms of positive eigenvalues of H

- Relation between bifurcations of resonances in JH and H

- What if continuous spectrum of H is sign-indefinite?

- What if J is not invertible?

- Dynamics of nonlinear waves beyond the linearized system