Heteroclinic orbits for travelling kinks in difference and nonlocal wave equations

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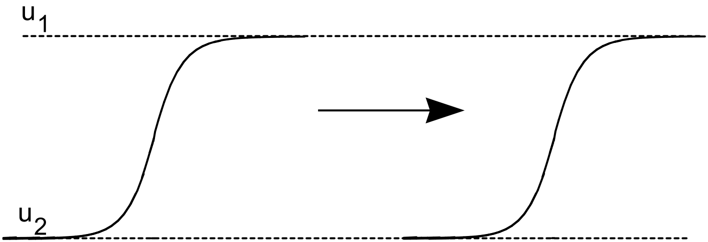
University of Pittsburg, November 20, 2015

1D case:

$u_{tt} - u_{xx} + V'(u) = 0$
where $V(u)$ is nonlinear potential (depends on a physical context)

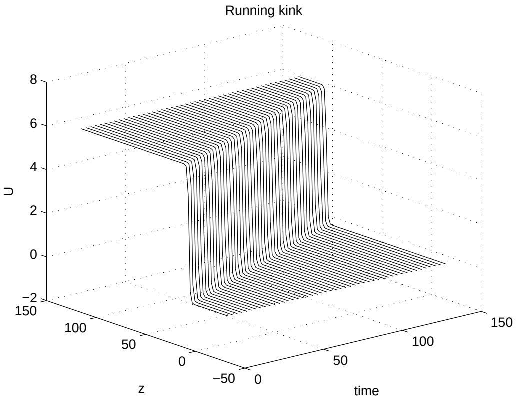
Kink (domain walls) solutions (steady or moving):

$\lim_{x\to-\infty} u(x,t) = u_2, \quad \lim_{x\to\infty} u(x,t) = u_1;$



Travelling waves: $u(x,t) = u(x - ct) \equiv u(z)$.

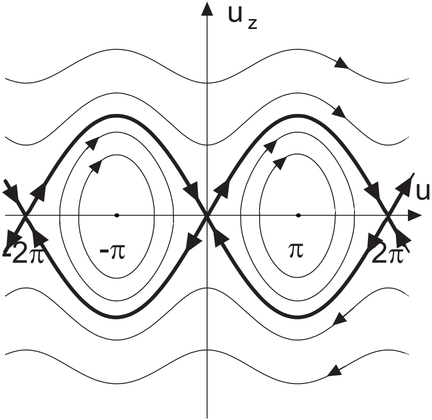
ODE: $(1 - c^2)u_{zz} - V'(u) = 0$



Example 1: the sine-Gordon equation

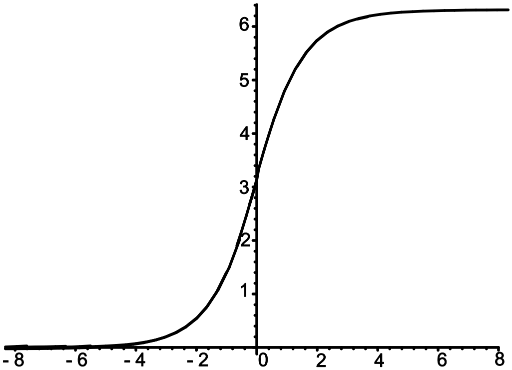
$u_{tt} - u_{xx} + \sin u = 0.$

Travelling waves: $(1 - c^2)u_{zz} = \sin u.$



- Only 2π-kink (antikink) solutions exist
- Solutions exist for arbitrary velocity c as long as $c^2 < 1$

$$u(z) = 4 \arctan \exp \left\{ \pm \frac{z - z_0}{\sqrt{1 - c^2}} \right\}, \quad z = x - ct.$$



Example 2: the double sine-Gordon equation

$u_{tt} - u_{xx} + \sin u - 2A \sin 2u = 0.$

- Exact 2π-kink solution exist for $1 - 4A > 0$:

$$u(z) = \pi + 2 \arctan \left(\frac{\sinh(\sqrt{1-4A}\,(z - z_0))}{\sqrt{1-4A}\sqrt{1-c^2}} \right),$$
$$z = x - ct$$

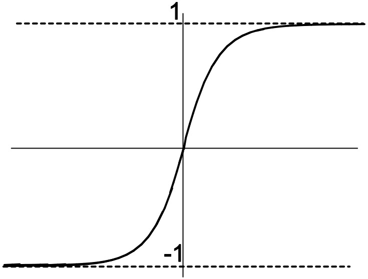
- Solution exist for arbitrary velocity c as long as $c^2 < 1$

Example 3: the ϕ^4 equation

$u_{tt} - u_{xx} - u + u^3 = 0.$

- Exact kink solution, exists for any $c^2 < 1$,

$$u(z) = \tanh\left(\frac{z - z_0}{\sqrt{2}\sqrt{1 - c^2}}\right), \quad z = x - ct$$



Example 4: the $\phi^4 - \phi^6$ equation

$u_{tt} - u_{xx} - u(1 - u^2)(1 + \gamma u^2) = 0.$

- Exact kink solution, exists for any $c^2 < 1$ and $\gamma > -1$:,

$$u(z) = \frac{\sqrt{18 + 6\gamma}\tanh\left(\frac{1}{2}\sqrt{2(1+\gamma)}\,(z - z_0)\right)}{\sqrt{18(1+\gamma) - 12\gamma\tanh^2\left(\frac{1}{2}\sqrt{2(1+\gamma)}\,(z - z_0)\right)}},$$

$$z = \frac{x - ct}{\sqrt{1 - c^2}}$$

Generic form:

$u_{tt} - \mathcal{L}u + V'(u) = 0$

- $\mathcal{L}$ is Fourier multiplier operator: $\widehat{\mathcal{L}u}(k) = P(k)\hat{u}(k)$;
- $P(k)$ is the symbol of the operator $\mathcal{L}$;
- If $P(k) = -k^2$, we are back to the nonlinear wave equation.

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Applications of nonlocal wave equations:

- discrete models (e.g. lattice models of solid state physics);
- complex dispersion (e.g. nonlinear optics);
- long-range interaction (e.g. models in solid state physics);
- specific geometry (e.g. Josephson junction theory).

Symbols:

- $P(k) = -\frac{4}{\lambda^2}\sin^2\left(\frac{\lambda k}{2}\right)$ (Frenkel-Kontorova model, solid state physics);
- $P(k) = -\frac{k^2}{1+\lambda^2 k^2}$ (Kac-Baker model, spin systems);
- $P(k) = -\frac{k^2}{\sqrt{1+\lambda^2 k^2}}$ (Silin-Gurevich model, Josephson junctions);

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In all these cases: $P(k) \equiv P_\lambda(k)$ depends on λ and

$$P_\lambda(k) \to -k^2 \quad \text{as} \quad \lambda \to 0.$$

As $\lambda \to 0$

$u_{tt} - \mathcal{L}_\lambda u + V'(u) = 0 \quad \Rightarrow \quad u_{tt} - u_{xx} + V'(u) = 0$

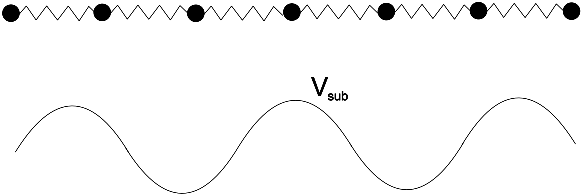
Main question:

What happens with kink solutions when switching from local case $\lambda = 0$ to nonlocal case $\lambda \neq 0$?

Example 5: the Frenkel-Kontorova model (1938)

$u_{tt}(x,t) - \frac{1}{\lambda^2}(u(x+\lambda,t) - 2u(x,t) + u(x-\lambda,t)) + \sin u(x,t) = 0.$

describes a chain of particles with nearest-neighbours interactions.



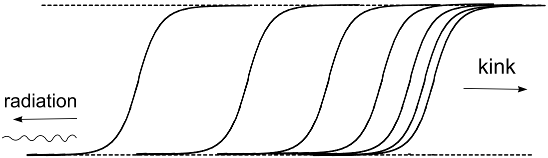
λ - a parameter of interaction between neighbours.

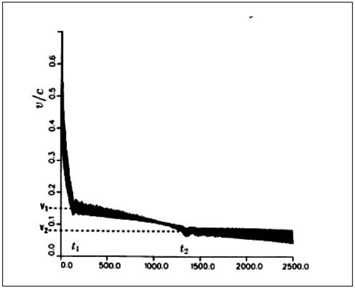
The symbol: $P(k) = -\frac{4}{\lambda^2}\sin^2\left(\frac{\lambda k}{2}\right)$

The results (well-known):

- There are at rest 2π-kinks (on-site and inter-site) in this model.
- No travelling 2π-kinks in this model.
- Infinitely many travelling 4π-kinks in this model.
- A kink-like excitation launched at some nonzero velocity emits radiation, slows down, and eventually stops.

The Frenkel-Kontorova model





(from M.Peyrard, M.D.Kruskal, Physica D, 14, p.88 (1984), initial velocity =0.8.)

Why do kink solutions disappear?

Consider linearized version of the Frenkel-Kontorova model at zero equilibrium:

$$u_{tt}(x,t) - \tfrac{1}{\lambda^2}(u(x+\lambda,t) - 2u(x,t) + u(x-\lambda,t)) + u(x,t) = 0.$$

Dispersion relation for Fourier transform:

$$1 + \frac{4}{\lambda^2}\sin^2\left(\frac{\lambda k}{2}\right) = c^2k^2, \quad k \in \mathbb{R},$$

For every $c \neq 0$, there exists at least one pair of solutions at $k = \pm k_0$.

SG equation with Kac-Baker interactions

Example 6: the sine-Gordon model with Kac-Baker interactions

$$u_{tt} - \frac{1}{2\lambda}\frac{d}{dx}\int_{-\infty}^{\infty} \exp\left(\frac{|x-x'|}{\lambda}\right) u_{x'}(x',t)\ dx' + \sin u = 0.$$

SG equation with Kac-Baker interactions

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The trick:

$$q(x,t) = \frac{1}{2\lambda}\ \int_{-\infty}^{+\infty} \exp\left\{-\frac{|x-x'|}{\lambda}\right\} u_{x'}(x',t)dx'$$

Then $q(x,t)$ is a solution of:

$$-\lambda^2 q_{xx} + q = u_x.$$

The symbol: $P(k) = -\frac{k^2}{1+\lambda^2 k^2}$

SG equation with Kac-Baker interactions

Travelling waves: $u(z) = u(x - ct)$

$$\boxed{\begin{aligned} & c^2 u_{zz} + \sin u = q_z \\ & -\lambda^2 q_{zz} + q = u_z \end{aligned}}$$

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Phase space: $\{u \pmod{2\pi}, u', q, q'\}$

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Phase space: $\{u \pmod{2\pi}, u', q, q'\}$

Equilibrium points:
$O_0(u = u' = q = q' = 0)$, $O_\pi(u = \pi, u' = q = q' = 0)$

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O_0 is the *saddle–center point*:

$$1 + \frac{k^2}{1+\lambda^2 k^2} = c^2 k^2$$

For every $c \neq 0$, there exists exactly one pair of solutions at $k = \pm k_0$.

SG equation with Kac-Baker interactions

Results:

- There are static 2π-kinks for $0 < \lambda < 1$.
- No travelling 2π-kinks in this model;
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- There are static 2π-kinks for $0 < \lambda < 1$.
- No travelling 2π-kinks in this model;
- Infinitely many 4π-kinks for discrete set of velocities;

Summary: switching from $\lambda = 0$ to $\lambda \neq 0$ results in disappearance of 2π-kink solutions in classical models.

Is this the only scenario?

Main Claim

Consider the bifurcation problem in the general form

$$L_\lambda u = F(u).$$

- L_λ - a Fourier multiplier operator with an even symbol $P_\lambda(k)$ such that

$$L_\lambda \to \frac{d^2}{dx^2} \text{ as } \lambda \to 0;$$

- $F(u)$ - an odd function such that $F(u_+) = F(u_-) = 0$ with $u_+ = -u_-$ and

$$F'(u_+) = F'(u_-) > 0$$

- Dispersion equation $P_\lambda(k) = F'(u_\pm)$ has one pair of roots $k = \pm k_0(\lambda)$, such that $k_0(\lambda) \to \infty$ as $\lambda \to 0$.

Let us consider the limiting equation $u''(z) = F(u(z))$ and assume:

- It has an odd kink solution $u_0(z)$ for $z \in \mathbb{R}$ such that $u_0(z) \to u_\pm$ as $z \to \pm\infty$.

- When $u_0(z)$ is continued for $z \in \mathbb{C}$, the closest to real axis singularities are located in quartets, e.g. in the upper half-plane at $z_\pm = \pm\alpha + i\beta$, $\alpha, \beta > 0$.

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There exists an infinite set of values $\{\lambda_n\}_{n\in\mathbb{N}}$, such that for each λ_n, the nonlinear equation $L_{\lambda_n} u = F(u)$ admits a kink solution. Moreover, the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfies the asymptotic law

$$k_0(\lambda_n) \sim (n\pi + \varphi_0)/\alpha, \quad n \to \infty,$$

where φ_0 is uniquely defined constant. Hence, $\lambda_n \to 0$ as $n \to \infty$.

Perturbation $v(z) = u(z) - u_0(z)$ satisfies the expanded equation

$$(L_\lambda - F'(u_0))\, v = H_\lambda + N(v),$$

where H_λ is explicitly computed from u_0 and $N(v)$ is $\mathcal{O}(v^2)$.

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- To satisfy the solvability condition at the leading order, we set

$$J_\pm(\lambda) := \int_{-\infty}^{\infty} e^{\pm i k(\lambda) z} H_\lambda(z) dz = 0$$

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- By Darboux principle and asymptotic analysis (Murray, 1984), if $H_\lambda(z) \sim C_0 \lambda^q e^{i\pi\kappa/2} (z - z_\pm)^\kappa$, then

$$J_\pm(\lambda) \sim \frac{4\pi\lambda^q |C_0| e^{-\beta k(\lambda)}}{\Gamma(-\kappa) |k(\lambda)|^{\kappa+1}} \cos(\alpha k(\lambda) + \pi/2 - \arg(C_0)).$$

Example 7: nonlocal double sine-Gordon model

$$u_{tt} - \frac{1}{2\lambda}\frac{d}{dx}\int_{-\infty}^{\infty} \exp\left(\frac{|x-x'|}{\lambda}\right) u_{x'}(x')\, dx' = \sin(u) + 2a\sin(2u).$$

Refs: Phys. Rev. Lett. **112**, 054103 (2014); Physica D **282**, 16 (2014)

- As $\lambda \to 0$, the 2π-kinks are given by:

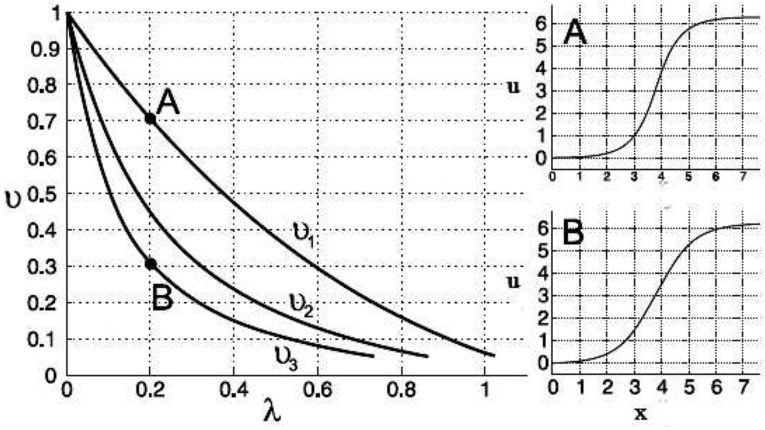
$$u_0(z) = \pi + 2\arctan\left[\frac{1}{\sqrt{1+4a}}\sinh\left(\frac{\sqrt{1+4a}}{\sqrt{1-c^2}}z\right)\right].$$

- Symmetric pairs of singularities exist for $a > 0$ at $z_\pm = \pm\alpha + i\beta$:

$$\alpha = \frac{\sqrt{1-c^2}}{2\sqrt{1+4a}}\cosh^{-1}(1+8a), \quad \beta = \frac{\pi\sqrt{1-c^2}}{2\sqrt{1+4a}}.$$

- For fixed $a > 0$, there exist a discrete set of curve in the (c, λ) plane, along which the 2π-kinks exist.

Nonlocal double SG model



Curves $c(\lambda)$ for $a = 1/8$.

The asymptotic law as $n \to \infty$:

$$2\alpha k_0(\lambda_n) \sim \pi(1+2n), \quad \Rightarrow \quad \pi(1+2n)\lambda_n = \delta(a, c),$$

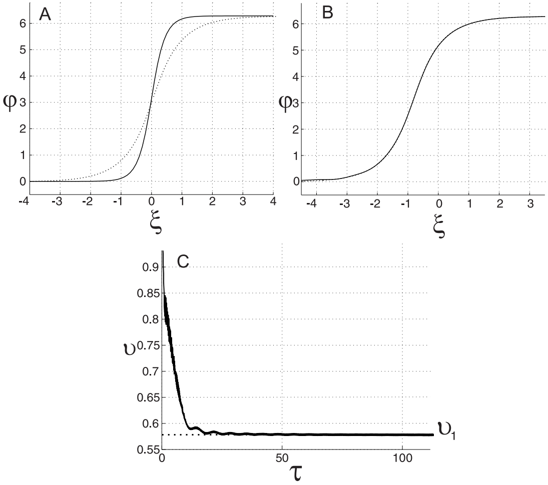
with $\varphi_0 = \pi/2$.

$1+2n$	1	3	5	7	9	11
$\delta/(\pi\lambda_n)$	3.7168	4.9763	6.3699	7.8595	9.4541	11.1396

Table: The values of $\delta/(\pi\lambda_n)$ for $a = 1/8$ and $c = 0.1$.

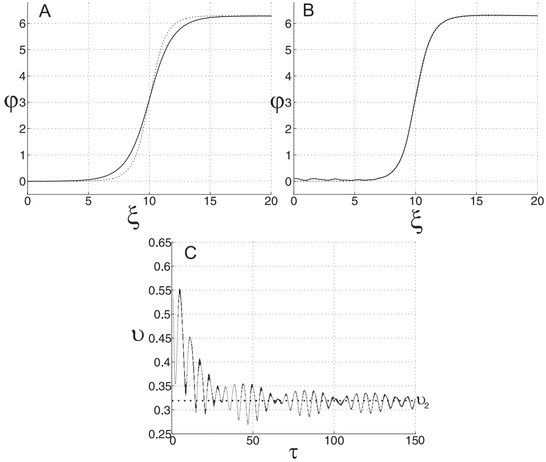
Nonlocal double SG model

Stability experiment 1



Evolution of kink-like excitation (high energy).

Stability experiment 2



Evolution of kink-like excitation (low energy).

Example 8: discrete ϕ^4-ϕ^6 model

$u_{tt} - \lambda^{-2}(u(x+\lambda) - 2u(x) + u(x-\lambda)) + u(1-u^2)(1+\gamma u^2) = 0.$

Refs: Phys. Rev. Lett. **112**, 054103 (2014)

- As $\lambda \to 0$, the kinks are given by:

$$u_0(z) = \frac{\sqrt{3+\gamma}\ \tanh(\eta z)}{\sqrt{3(1+\gamma) - 2\gamma\ \tanh^2(\eta z)}},\ \eta = \frac{\sqrt{1+\gamma}}{\sqrt{2(1-c^2)}}.$$

- Symmetric pairs of singularities exist for $\gamma > 0$ at $z_\pm = \pm\alpha + i\beta$:

$$\alpha = \frac{\sqrt{1-c^2}}{2\sqrt{1+\gamma}} \cosh^{-1}\left(\frac{3+5\gamma}{3+\gamma}\right), \quad \beta = \frac{\pi\sqrt{1-c^2}}{\sqrt{2(1+a)}}.$$

- For fixed $\gamma > 0$, there exist a discrete set of curve in the (c, λ) plane, along which the kinks exist.

The asymptotic law as $n \to \infty$:

$$4\alpha k_0(\lambda_n) \sim \pi(3+4n), \quad \Rightarrow \quad \pi(3+4n)\lambda_n = \chi(\gamma, c),$$

with $\varphi_0 = 3\pi/4$.

$3+4n$	3	7	11	15
$\chi/(\pi\lambda_n)$	3.5303	7.3547	11.1520	15.0329

Table: The values of $\chi/(\pi\lambda_n)$ for $\gamma = 5$ and $c = 0.6$.

Example 9: discrete ϕ^4 model

$u_{tt} - \lambda^{-2}(u(x+\lambda) - 2u(x) + u(x-\lambda)) + u(x)\left(1 - u(x)^2\right) = 0.$

Refs: Nonlinearity **19**, 217 (2006)

- As $\lambda \to 0$, the kinks are given by:

$$u_0(z) = \tanh(\eta z),\ \eta = \frac{1}{2\sqrt{1-c^2}}.$$

- Singularity exists at $z = i\pi\sqrt{1-c^2}$.
- No kinks exist for any $c \neq 0$.

Example 10: another discrete ϕ^4 model

$$u_{tt} - \lambda^{-2}(u(x+\lambda) - 2u(x) + u(x-\lambda))$$
$$+\frac{1}{2}(u(x+\lambda) + u(x-\lambda))\left(1 - \frac{1}{2}u(x+\lambda)^2 - \frac{1}{2}u(x-\lambda)^2\right) = 0.$$

Refs: Nonlinearity **19**, 217 (2006)

- As $\lambda \to 0$, the kinks are still given by:

$$u_0(z) = \tanh(\eta z),\ \eta = \frac{1}{2\sqrt{1-c^2}}.$$

- Singularity exists at $z = i\pi\sqrt{1-c^2}$.
- Three moving kinks exist for three values of $c \neq 0$ at fixed $\lambda \neq 0$.

Conclusion

Summary: in Examples 7-8, switching from $\lambda = 0$ to $\lambda \neq 0$ results in selecting a countable set of velocities for radiationless kink propagation.

• The first ideas about existence of such countable sets go back to the works of V.G. Gelfreich (1990,2008).

• No analytical proof of the main claim exists for now.

• It has been checked for several other models: triple sine-Gordon model, fifth-order Korteweg-de Vries equation, saturable discrete nonlinear Schrödinger equation, ...

• Apparently, it applies to more sophisticated examples, such as diatomic Toda lattice