Rogue waves on the periodic background

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The rogue wave of the cubic NLS equation

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits the exact solution

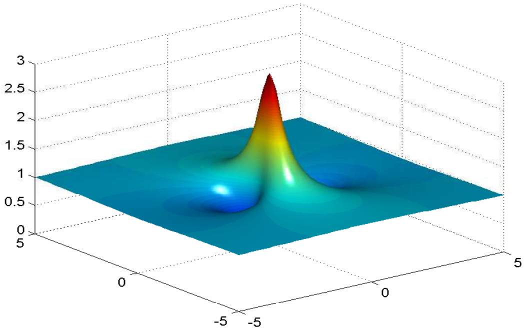
$$\psi(x,t) = \left[1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right] e^{it}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.

Properties of the rogue wave:

- It is related to modulational instability of CW background $\psi_0(x,t) = e^{it}$.
- It comes from nowhere: $|\psi(x,t)| \to 1$ as $|x| + |t| \to \infty$.
- It is magnified at the center: $M_0 := |\psi(0,0)| = 3$.

The rogue wave of the cubic NLS equation



Possible developments:

- To generate higher-order rational solutions for multiple rogue waves...
- To extend constructions in other basic integrable PDEs...

Periodic wave background

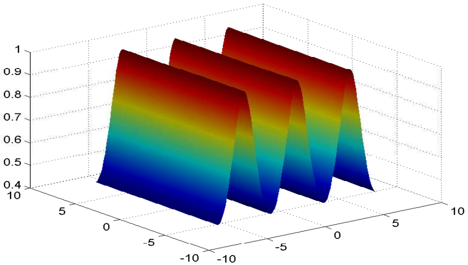
The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits other wave solutions, e.g. the periodic waves of trivial phase

$$\psi_{\rm dn}(x,t) = {\rm dn}(x;k)e^{i(1-k^2/2)t}, \quad \psi_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{i(k^2-1/2)t}$$

where $k \in (0,1)$ is elliptic modulus.



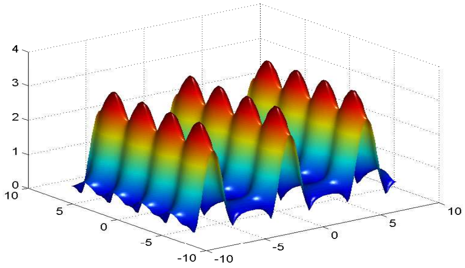
Double-periodic wave background

Double-periodic solutions (Akhmediev, Eleonskii, Kulagin, 1987):

$$\psi(x,t) = k\frac{\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa) + i\sqrt{1+k}\mathrm{sn}(t;k)\mathrm{dn}(\sqrt{1+k}x;\kappa)}{\sqrt{1+k}\mathrm{dn}(\sqrt{1+k}x;\kappa) - \mathrm{dn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa)}e^{it},$$

$$\psi(x,t) = \frac{\mathrm{dn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa) + i\sqrt{k(1+k)}\mathrm{sn}(t;k)}{\sqrt{1+k} - \sqrt{k}\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)}e^{ikt}, \quad \kappa = \frac{\sqrt{1-k}}{\sqrt{2}}.$$

where $k \in (0,1)$ is elliptic modulus.



Main question

Can we obtain a rogue wave on the background ψ_0 such that

$$\inf_{x_0, t_0, \alpha_0 \in \mathbb{R}} \sup_{x \in \mathbb{R}} \left| \psi(x,t) - \psi_0(x - x_0, t - t_0) e^{i\alpha_0} \right| \to 0 \quad \text{as} \quad t \to \pm\infty \quad ???$$

This rogue wave *appears from nowhere and disappears without trace*.

Further questions:

- Magnification factors for rogue waves
- Spectral representation and inverse scattering
- Robustness (stability) in the time evolution.
- Extensions to quasi-periodic background.
- Extensions to multi-soliton background.

Darboux transformation as the main tool

Let u be a solution of the NLS. It is a potential of the compatible Lax system

$$\varphi_x = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$$

and

$$\varphi_t = V(\lambda, u)\varphi, \qquad V(\lambda, u) = i \begin{pmatrix} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda\bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{pmatrix},$$

so that $\varphi_{xt} = \varphi_{tx}$.

Let $\varphi = (p_1, q_1)$ be a nonzero solution of the Lax system for $\lambda = \lambda_1 \in \mathbb{C}$. The following one-fold Darboux transformation (DT):

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2},$$

provides another solution $\hat{u}$ of the same NLS equation.

Preliminary literature

- Numerical computations of eigenfunctions for DT on *dn*-, *cn*-, and double-periodic backgrounds:
  (Kedziora–Ankiewicz–Akhmediev, 2014) (Calini–Schober, 2017)
- Emergence of rogue waves in simulations of modulation instability of *dn*-periodic waves:
  (Agafontsev–Zakharov, 2016)
- Magnification factors of quasi-periodic solutions from analysis of Riemann's Theta functions:
  (Bertola–Tovbis, 2017) (Wright, 2019)
- Rogue waves from superpositions of nearly identical solitons:
  (Bilman–Buckingham, 2018) (Slunyaev, 2019)

Algebraic method - Step 1

Consider the spectral problem

$$\varphi_x = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$$

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with $\varphi = (p_1, q_1) \in \mathbb{C}^2$ and set

$$\begin{cases} u = p_1^2 + \bar{q}_1^2, \\ \bar{u} = \bar{p}_1^2 + q_1^2. \end{cases}$$

The spectral problem becomes the Hamiltonian system of degree two generated by the Hamiltonian function

$$H = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2}(p_1^2 + \bar{q}_1^2)(\bar{p}_1^2 + q_1^2).$$

The algebraic technique is called the “nonlinearization” of Lax pair (Cao–Geng, 1990) (Cao–Wu–Geng, 1999) (R.Zhou, 2009)

Hamiltonian system and constraints

The Hamiltonian system is integrable with two constants of motion:

$$\begin{aligned} H &= \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2}(p_1^2 + \bar{q}_1^2)(\bar{p}_1^2 + q_1^2), \\ F &= i(p_1 q_1 - \bar{p}_1 \bar{q}_1). \end{aligned}$$

The constraints between u and (p_1, q_1) are extended as:

$$\begin{aligned} u &= p_1^2 + \bar{q}_1^2, \\ \frac{du}{dx} + 2iFu &= 2(\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2), \\ \frac{d^2u}{dx^2} + 2|u|^2 u + 2iF\frac{du}{dx} - 4Hu &= 4(\lambda_1^2 p_1^2 + \bar{\lambda}_1^2 \bar{q}_1^2). \end{aligned}$$

Compatible potentials $u(x)$ satisfy the closed second-order ODE:

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0,$$

where $c := F + i(\lambda_1 - \bar{\lambda}_1)$ and $b := H + iF(\lambda_1 - \bar{\lambda}_1) + |\lambda_1|^2$.

Integrability of the Hamiltonian system

The Hamiltonian system is a compatibility condition of the Lax equation

$$\frac{d}{dx}W(\lambda) = U(\lambda,u)W(\lambda) - W(\lambda)U(\lambda,u),$$

where $U(\lambda,u)$ is the same as in the Lax system and

$$W(\lambda) = \left(\begin{array}{cc} W_{11}(\lambda) & W_{12}(\lambda) \\ \bar{W}_{12}(-\lambda) & -\bar{W}_{11}(-\lambda) \end{array} \right),$$

with

$$\begin{aligned} W_{11}(\lambda) &= 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{\bar{p}_1 \bar{q}_1}{\lambda + \bar{\lambda}_1}, \\ W_{12}(\lambda) &= \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \bar{\lambda}_1}. \end{aligned}$$

Simple algebra shows

$$W_{11}(\lambda) = \frac{\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2}{(\lambda-\lambda_1)(\lambda+\bar{\lambda}_1)}, \quad W_{12}(\lambda) = \frac{u\lambda + icu + \frac{1}{2}u'}{(\lambda-\lambda_1)(\lambda+\bar{\lambda}_1)}.$$

Closure relations

The $(1,2)$-element of the Lax equation,

$$\frac{d}{dx} W_{12}(\lambda) = 2\lambda W_{12}(\lambda) - 2u W_{11}(\lambda),$$

yields the second-order equation on u:

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0.$$

$\det W(\lambda)$ is constant in (x,t) and has simple poles at λ_1 and $-\bar{\lambda}_1$:

$$\det[W(\lambda)] = -[W_{11}(\lambda)]^2 - W_{12}(\lambda)\bar{W}_{12}(-\lambda) = -\frac{P(\lambda)}{(\lambda-\lambda_1)^2(\lambda+\bar{\lambda}_1)^2}$$

so that $P(\lambda)$ is constant in (x,t) and has roots at λ_1 and $-\bar{\lambda}_1$:

$$P(\lambda) = (\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2)^2 - (u\lambda + icu + \frac{1}{2}u')(\bar{u}\lambda + ic\bar{u} - \frac{1}{2}\bar{u}')$$

Conserved quantities

The second-order equation on u

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0$$

is now closed with the conserved quantities

$$\left. \begin{array}{r} i(u'\bar{u} - u\bar{u}') - 2c|u|^2 = 4a, \\ |u'|^2 + |u|^4 + 4b|u|^2 = 8d. \end{array} \right\}$$

These equations describe a general class of **traveling wave solutions**:

$$\psi(x,t) = u(x+ct)e^{-2ibt}.$$

The polynomial $P(\lambda)$ in $\det W(\lambda)$ is given by

$$P(\lambda) = \lambda^4 + 2ic\lambda^3 + (2b - c^2)\lambda^2 + 2i(a+bc)\lambda + b^2 - 2ac + 2d,$$

with roots at λ_1 and $-\bar{\lambda}_1$. (Another pair also exists.)

Periodic waves of trivial phase

For traveling wave solutions:

- $c = 0$ can be set without loss of generality.
- $a = 0$ is set for waves with trivial phase.

The real function $u(x)$ is determined by the quadrature:

$$\left(\frac{du}{dx}\right)^2 + u^4 + 4bu^2 = 8d$$

with two parameters b,d. Parameterizing $V(u) = u^4 + 4bu^2 - 8d$ by two pairs of roots:

$$\begin{cases} -4b = u_1^2 + u_2^2, \\ -8d = u_1^2 u_2^2. \end{cases}$$

we get **two families of traveling wave solutions:**

- $0 < u_2 < u_1$: $u(x) = u_1 \mathrm{dn}(u_1 x; k)$
- $u_2 = i\nu_2$: $u(x) = u_1 \mathrm{cn}(\alpha x; k)$, $\alpha = \sqrt{u_1^2 + \nu_2^2}$

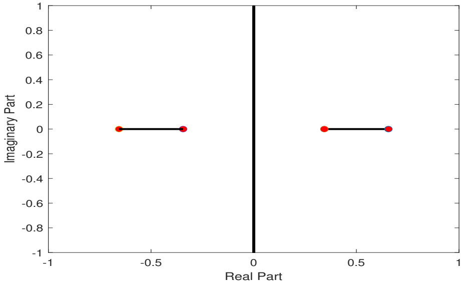
Lax spectrum of dn-periodic waves

Polynomial $P(\lambda)$ simplifies in terms of the turning points u_1, u_2:

$$P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2$$

with two pairs of roots

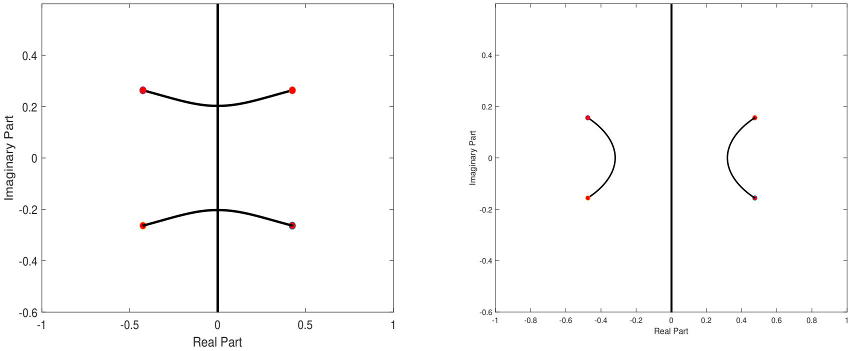
$$\lambda_1^{\pm} = \pm\frac{u_1 + u_2}{2}, \quad \lambda_2^{\pm} = \pm\frac{u_1 - u_2}{2}.$$



Lax spectrum of cn-periodic waves

If $u_2 = i\nu_2$, there is one quadruplet of roots:

$$\lambda_1^{\pm} = \pm \frac{u_1 + i\nu_2}{2}, \quad \lambda_2^{\pm} = \pm \frac{u_1 - i\nu_2}{2}.$$

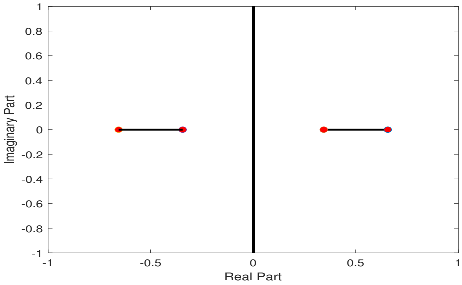


En route to rogue waves

Let $\varphi = (p_1, q_1)$ be a nonzero solution of the Lax system for $\lambda = \lambda_1 \in \mathbb{C}$. The one-fold Darboux transformation

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1) p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2},$$

gives another solution $\hat{u}$ of the same NLS equation.



Question: which value of λ_1 to use?

Algebraic method - Step 2

Evaluating the matrix elements at simple poles λ_1 and $-\bar{\lambda}_1$

$$
\begin{aligned}
W_{11}(\lambda) &= 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{\bar{p}_1 \bar{q}_1}{\lambda + \bar{\lambda}_1} = \frac{\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)}, \\
W_{12}(\lambda) &= \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \bar{\lambda}_1} = \frac{u\lambda + icu + \frac{1}{2}u'}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)},
\end{aligned}
$$

we can derive the inverse relations between the potential u and the squared eigenfunctions:

$$
\begin{aligned}
p_1^2 &= \frac{1}{\lambda_1 + \bar{\lambda}_1} \left(\frac{1}{2} u' + icu + \lambda_1 u \right), \\
q_1^2 &= \frac{1}{\lambda_1 + \bar{\lambda}_1} \left(-\frac{1}{2} u' + icu + \lambda_1 u \right), \\
p_1 q_1 &= -\frac{1}{\lambda_1 + \bar{\lambda}_1} \left(b + \frac{1}{2}|u|^2 + i\lambda_1 c + \lambda_1^2 \right).
\end{aligned}
$$

The eigenfunction $\varphi = (p_1, q_1)$ is periodic if u is periodic.

Second linearly independent solution

Let us define the second solution $\varphi = (\hat{p}_1, \hat{q}_1)$ by

$$\hat{p}_1 = p_1 \phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1 \phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

such that $p_1 \hat{q}_1 - \hat{p}_1 q_1 = 2$ (Wronskian is constant). Then, scalar function $\phi_1(x,t)$ satisfies

$$\frac{\partial \phi_1}{\partial x} = -\frac{4(\lambda_1 + \bar{\lambda}_1)\bar{p}_1 \bar{q}_1}{(|p_1|^2 + |q_1|^2)^2}$$

and

$$\frac{\partial \phi_1}{\partial t} = -\frac{4i(\lambda_1^2 - \bar{\lambda}_1^2)\bar{p}_1 \bar{q}_1}{(|p_1|^2 + |q_1|^2)^2} + \frac{2i(\lambda_1 + \bar{\lambda}_1)(u\bar{p}_1^2 + \bar{u}\bar{q}_1^2)}{(|p_1|^2 + |q_1|^2)^2}.$$

The system is compatible as it is obtained from Lax equation.

Second solutions for periodic waves

For periodic waves with the trivial phase, variables are separated by

$$u(x,t) = U(x)e^{-2ibt}, \quad p_1(x,t) = P_1(x)e^{-ibt}, \quad q_1(x,t) = Q_1(x)e^{ibt},$$

where U is real, either $U(x) = \mathrm{dn}(x;k)$ or $U(x) = k\,\mathrm{cn}(x;k)$, whereas $|p_1|^2 + |q_1|^2 = \mathrm{dn}(x;k)$ in both cases.

Integrating linear equations for $\phi_1(x,t)$ yields

$$\phi_1(x,t) = 2x + 2i(1 \pm \sqrt{1-k^2})t \pm 2\sqrt{1-k^2}\int_0^x \frac{dy}{\mathrm{dn}^2(y;k)}$$

and

$$\phi_1(x,t) = 2k^2 \int_0^x \frac{\mathrm{cn}^2(y;k)dy}{\mathrm{dn}^2(y;k)} \mp 2ik\sqrt{1-k^2}\int_0^x \frac{dy}{\mathrm{dn}^2(y;k)} + 2ikt$$

from which it is obvious that $|\phi_1| \to \infty$ as $t \to \pm\infty$.

Algebraic method - Step 3

Rogue waves on the background u are generated by the DT:

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1 \bar{\hat{q}}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2},$$

where

$$\hat{p}_1 = p_1\phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1\phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

As $t \to \pm\infty$,

$$\hat{u}(x,t)|_{|\phi_1|\to\infty} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2}$$

which is a translation of the periodic wave u, e.g.

$$\hat{u}(x,t)|_{|\phi_1|\to\infty} = \frac{\sqrt{1-k^2}}{\mathrm{dn}(x;k)} = \mathrm{dn}(x + K(k);k)$$

or

$$\hat{u}(x,t)|_{|\phi_1|\to\infty} = -\frac{k\sqrt{1-k^2}\mathrm{sn}(x;k)}{\mathrm{dn}(x;k)} = k\mathrm{cn}(x + K(k);k).$$

Magnification factor

Rogue waves on the background u are generated by the DT:

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1 \bar{\hat{q}}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2},$$

where

$$\hat{p}_1 = p_1 \phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1 \phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

At the center of the rogue wave,

$$\hat{u}(x,t)|_{\phi_1 = 0} = u - \frac{2(\lambda_1 + \bar{\lambda}_1) p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2} = 2u - \tilde{u},$$

hence the magnification factor does not exceed *three magnification* if the rogue solution is obtained by the one-fold transformation.

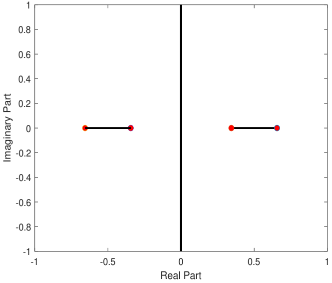
Rogue wave on the *dn*-periodic wave

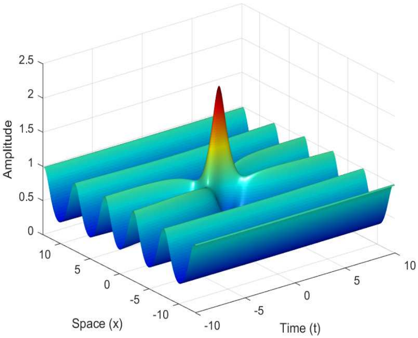
The *dn*-periodic wave is

$$u(x,t) = \mathrm{dn}(x;k)e^{i(1-k^2/2)t}$$

The rogue wave for the larger eigenvalue λ_1 has the larger magnification:

$$M(k) = 2 + \sqrt{1-k^2}, \quad k \in [0,1].$$





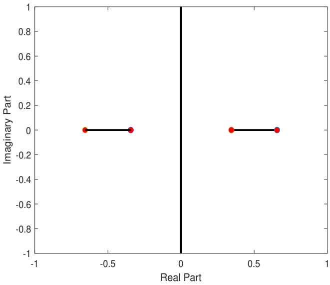
Another rogue wave on the *dn*-periodic wave

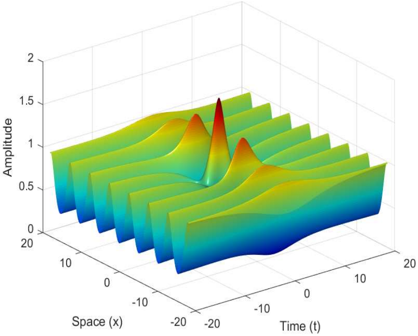
The *dn*-periodic wave is

$$u(x,t) = \mathrm{dn}(x;k)e^{i(1-k^2/2)t}$$

The rogue wave for the smaller eigenvalue λ_1 has the smaller magnification:

$$M(k) = 2 - \sqrt{1-k^2}, \quad k \in [0,1].$$





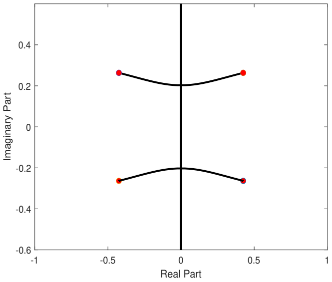
Rogue wave on the *cn*-periodic wave

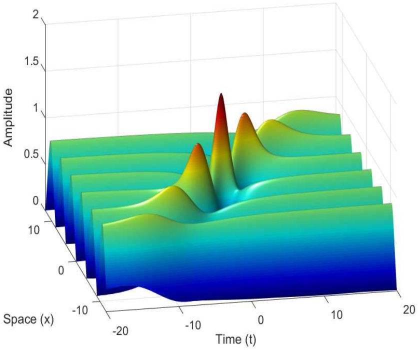
The *cn*-periodic wave is

$$\psi_{\rm cn}(x,t) = k\,{\rm cn}(x;k)e^{i(k^2-1/2)t}$$

The rogue wave has the exact magnification factor:

$$M(k) = 2, \quad k \in [0,1].$$





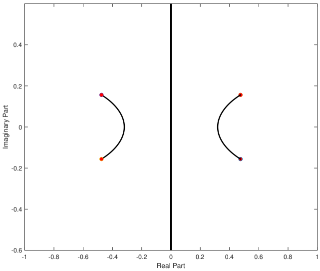
Rogue wave on the *cn*-periodic wave

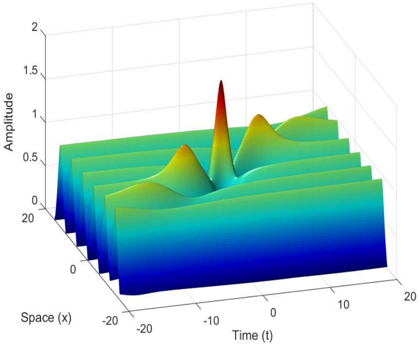
The *cn*-periodic wave is

$$\psi_{\mathrm{cn}}(x,t) = k\,\mathrm{cn}(x;k)e^{i(k^2-1/2)t}$$

The rogue wave has the exact magnification factor:

$$M(k) = 2, \quad k \in [0,1].$$





Relation to modulation instability of the periodic wave

Substituting $u(x,t) = e^{-2ibt}\left[U(x) + \tilde{u}(x,t)\right]$ into the NLS equation

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0$$

and linearizing at $\tilde{u}$ yields the linearized evolution problem

$$i\tilde{u}_t + \frac{1}{2}\tilde{u}_{xx} + 2(b + U^2)\tilde{u} + U^2\bar{\tilde{u}} = 0.$$

Since U does not depend on t, we can separate the variables $\tilde{u}(x,t) = \tilde{U}(x)e^{\Gamma t}$ and obtain the spectral problem for Γ:

$$\left[\begin{array}{cc} -\frac{1}{2}\partial_x^2 - 2(b+U^2) & -U^2 \\ -U^2 & -\frac{1}{2}\partial_x^2 - 2(b+U^2) \end{array}\right]\left[\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right] = i\Gamma \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right],$$

where

$$\tilde{U}(x+L) = e^{i\gamma L}\tilde{U}(x)$$

with Floquet parameter $\gamma \in [-\pi/L, \pi/L]$.

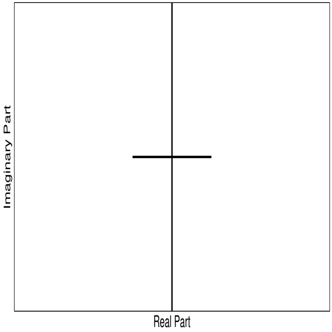
The periodic wave is modulationally unstable if $\exists \Gamma$ with $\mathrm{Re}(\Gamma) > 0$ for small γ.

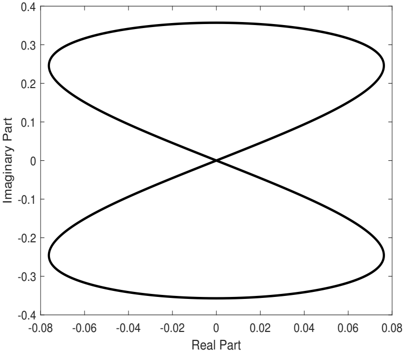
Relation to modulation instability of the periodic wave

If λ belongs to the Lax spectrum and $P(\lambda)$ is the polynomial in

$$P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2$$

then $\Gamma := \pm 2i\sqrt{P(\lambda)}$ is in the modulation instability spectrum. (Deconinck–Segal, 2017) (Deconinck–Upsal, 2019)





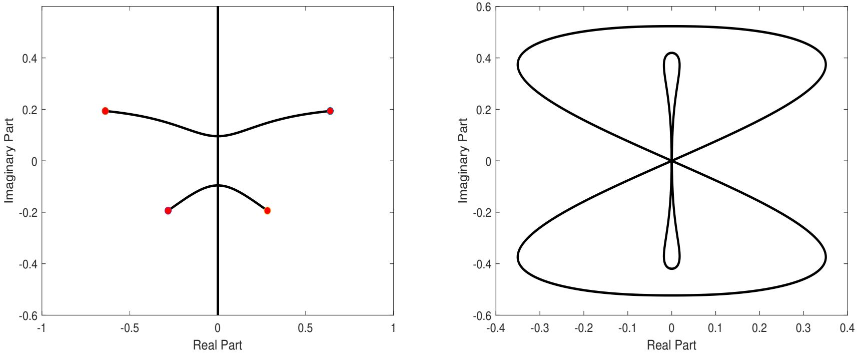
Relation to modulation instability of the periodic wave

Here is an example of the periodic wave with nontrivial phase

$$u(x) = R(x) e^{i\Theta(x)} e^{2ibt}$$

with

$$R(x) = \sqrt{\beta - k^2 \mathrm{sn}^2(x;k)}, \quad \Theta(x) = -2e \int_0^x \frac{dx}{R(x)^2}.$$



Relation to modulation instability of the periodic wave

Here is an example of the periodic wave with nontrivial phase

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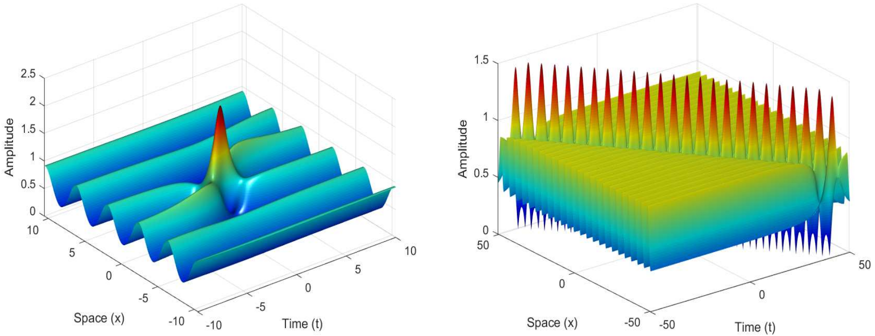


Parameter plane for periodic waves

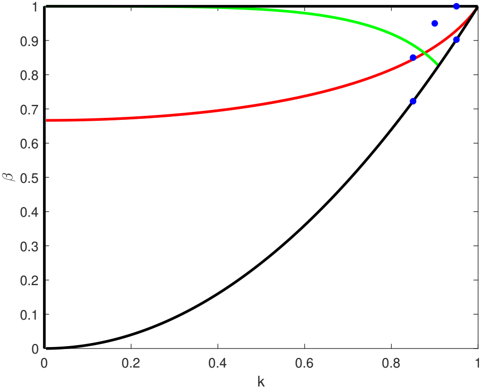


Figure: The black curves are boundaries of the triangular region where the periodic waves with nontrivial phase exist. The blue dots show parameter values of (β, k) for the solutions chosen for numerical illustrations. The red curve shows the curve where the rogue wave is not localized.

Algebraic method with two eigenvalues

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with $\varphi = (p_1, q_1) \in \mathbb{C}^2$ and $\lambda = \lambda_2 \in \mathbb{C}$ with $\varphi = (p_2, q_2) \in \mathbb{C}^2$ such that $\lambda_1 \neq \pm\lambda_2$ and $\lambda_1 \neq \pm\bar{\lambda}_2$. Set

$$u = p_1^2 + \bar{q}_1^2 + p_2^2 + \bar{q}_2^2.$$

The algebraic method produces the third-order equation

$$u''' + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0,$$

with three constants of motion:

$$\left.\begin{array}{r} d + \frac{1}{2}b|u|^2 + \frac{i}{4}c(u'\bar{u} - u\bar{u}') + \frac{1}{8}(u\bar{u}'' + u''\bar{u} - |u'|^2 + 3|u|^4) = 0, \\ 2e - a|u|^2 - \frac{1}{4}c(|u'|^2 + |u|^4) + \frac{i}{8}(u''\bar{u}' - u'\bar{u}'') = 0, \\ f - \frac{i}{2}a(u'\bar{u} - u\bar{u}') + \frac{1}{4}b(|u'|^2 + |u|^4) + \frac{1}{16}(|u'' + 2|u|^2 u|^2 - (u'\bar{u} - u\bar{u}')^2) = 0. \end{array}\right\}$$

Eigenvalues λ_1 and λ_2 are found among three roots of the polynomial

$$\begin{aligned} P(\lambda) &= \lambda^6 + 2ic\lambda^5 + (2b - c^2)\lambda^4 + 2i(a + bc)\lambda^3 + (b^2 - 2ac + 2d)\lambda^2 \\ &\quad + 2i(e + ab + cd)\lambda + f + 2bd - 2ce - a^2. \end{aligned}$$

Double-periodic solutions

Double-periodic solutions (Akhmediev, Eleonskii, Kulagin, 1987) correspond to $c = a = e = 0$. The solution takes the explicit form:

$$u(x,t) = [Q(x,t) + i\delta(t)]\, e^{i\theta(t)}.$$

where $Q(x,t)$ and $\delta(t)$ are found from the first-order quadratures:

$$\delta(t) = \frac{\sqrt{z_1 z_3}\,\mathrm{sn}(\mu t; k)}{\sqrt{z_3 - z_1 \mathrm{cn}^2(\mu t; k)}},$$

with $0 \leq z_1 \leq z_2 \leq z_3$ and

$$Q(x,t) = Q_4 + \frac{(Q_1 - Q_4)(Q_2 - Q_4)}{(Q_2 - Q_4) + (Q_1 - Q_2)\mathrm{sn}^2(\nu x; \kappa)},$$

with $Q_4 \leq Q_3 \leq Q_2 \leq Q_1$.

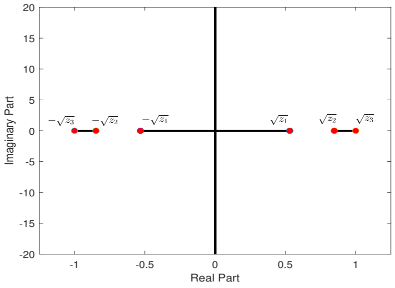
By construction, $\pm\sqrt{z_1}$, $\pm\sqrt{z_2}$, $\pm\sqrt{z_3}$ are roots of $P(\lambda)$:

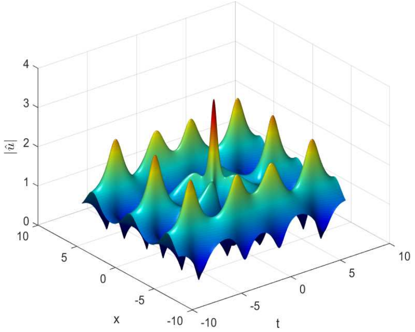
$$P(\lambda) = \lambda^6 + 2b\lambda^4 + (b^2 + 2d)\lambda^2 + f + 2bd.$$

Lax spectrum and rogue waves: real roots

The double-periodic solution if $z_{1,2,3}$ are real:

$$u(x,t) = k\frac{\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa) + i\sqrt{1+k}\mathrm{sn}(t;k)\mathrm{dn}(\sqrt{1+k}x;\kappa)}{\sqrt{1+k}\mathrm{dn}(\sqrt{1+k}x;\kappa) - \mathrm{dn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa)} e^{it},$$





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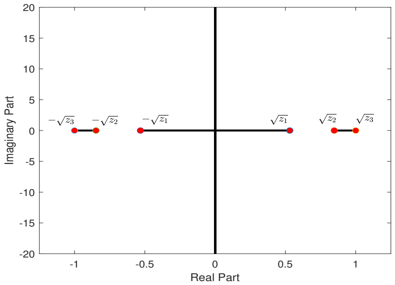

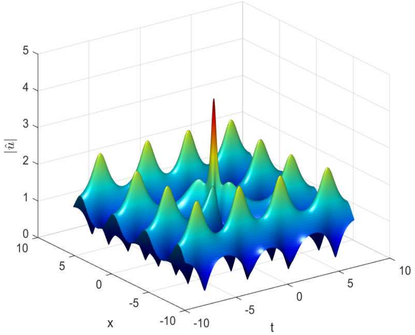

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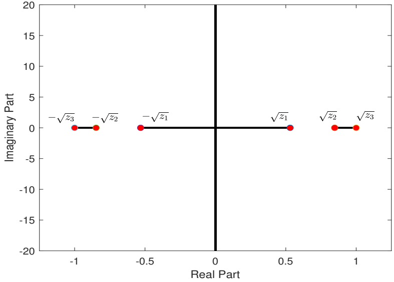
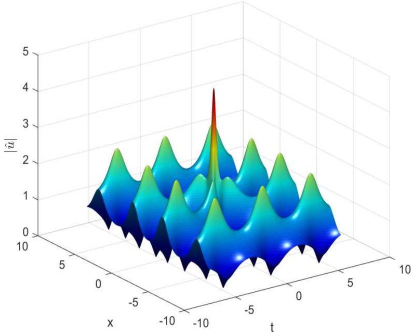
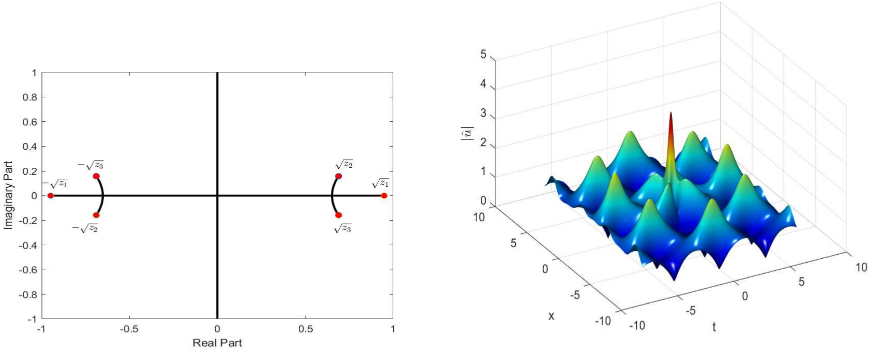




Lax spectrum and rogue waves: complex roots

The double-periodic solution if z_1 is real and $z_{2,3}$ are complex:

$$u(x,t) = \frac{\mathrm{dn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa) + i\sqrt{k(1+k)}\mathrm{sn}(t;k)}{\sqrt{1+k} - \sqrt{k}\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)} e^{ikt}, \quad \kappa = \frac{\sqrt{1-k}}{\sqrt{2}}.$$



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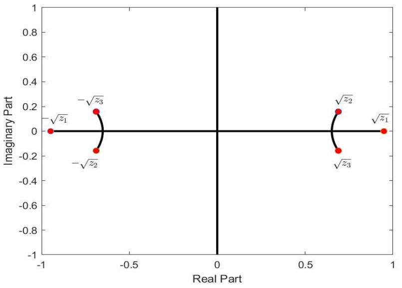

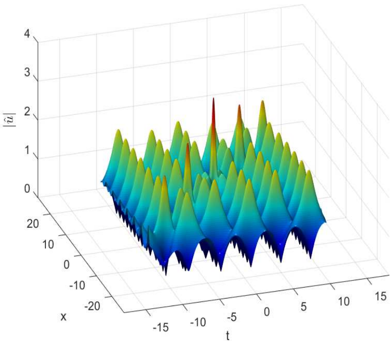

Magnification factors

Simplest definition of the magnification factor:

$$M_1 := \frac{\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|}{\max_{(x,t)\in\mathbb{R}^2} |u(x,t)|} \leq 3.$$

Definition of the magnification factor used in physics literature:

$$M_2 := \frac{\max_{(x,t)\in\mathbb{R}^2} |\hat{u}(x,t)|}{\mathrm{mean}_{(x,t)\in\mathbb{R}^2} |u(x,t)|}.$$

Rogue wave	Solution	M_1	M_2
$\lambda_1 = \sqrt{z_1}$	real roots	1.45	3.96
$\lambda_2 = \sqrt{z_2}$	same	1.71	4.68
$\lambda_3 = \sqrt{z_3}$	same	1.84	5.03
$\lambda_1 = \sqrt{z_1}$	complex roots	1.80	4.67
$\lambda_2 = \sqrt{\xi + i\eta}$	same	1.60	4.15

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- New method is developed for computations of eigenvalues and eigenfunctions of the Lax system for periodic and double-periodic waves.
- New exact solutions are obtained for rogue waves on the background of periodic and double-periodic waves.
- Magnification factor is computed exactly at the rogue waves.

Further directions:

- Characterize eigenvalues, eigenfunctions, and rogue waves on general quasi-periodic solutions.
- Observe rogue waves on the periodic background in water wave experiments (Amin Chabchoub, Sydney).

Thank you! Questions???

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