Spectral stability of periodic waves in the generalized reduced Ostrovsky equation

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- E.R. Johnson (University College London) - J. Diff. Eqs. (2016)
- A. Geyer (Delft University of Technology) - Lett. Math. Phys. (2017)

Ostrovsky equation for rotating fluid

The **Ostrovsky equation** is a model for small-amplitude long waves in a rotating fluid of a finite depth [Ostrovsky, 1978]:

$$(u_t + uu_x - \beta u_{xxx})_x = \gamma u,$$

where β and γ are real coefficients.

When $\beta = 0$ and $\gamma = 1$, the Ostrovsky equation is

$$(u_t + uu_x)_x = u,$$

and is known under the names of

- the short-wave equation [Hunter, 1990];
- Ostrovsky–Hunter equation [Boyd, 2005];
- reduced Ostrovsky equation [Stepanyants, 2006];
- the Vakhnenko equation [Vakhnenko & Parkes, 2002].

We will use the terminology of the **reduced Ostrovsky equation**.

Modified Ostrovsky equation

Internal waves are described by the modified Ostrovsky equation [R. Grimshaw et al., 1998]:

$$(u_t + u^2 u_x - \beta u_{xxx})_x = \gamma u.$$

When $\beta = 0$ and $\gamma = 1$, the modified Ostrovsky equation

$$(u_t + u^2 u_x)_x = u$$

has been studied by [E.R. Johnson, R. Grimshaw, 2014]

Note that the reduced modified Ostrovsky equation is different from the **short-pulse equation** derived as a model for propagation of ultra-short pulses with few cycles on the pulse scale [Schäfer, Wayne 2004]:

$$u_{xt} = u + \left(u^3\right)_{xx}.$$

Well-posedness results

Consider the generalized reduced Ostrovsky equation for an integer p:

$$(u_t + u^p u_x)_x = u.$$

We are interested in **travelling $2T$-periodic waves and their stability**.
All solutions satisfy the constraint $\int_{-T}^{T} u dx = 0$.
We denote the L^2 space of $2T$-periodic functions with zero mean by $\dot{L}^2_{\text{per}}$.

Well-posedness results

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We are interested in **travelling $2T$-periodic waves and their stability**.
All solutions satisfy the constraint $\int_{-T}^{T} u dx = 0$.
We denote the L^2 space of $2T$-periodic functions with zero mean by $\dot{L}^2_{\rm per}$.

- Local solutions exist in $\dot{H}^s_{\rm per}$ for $s > \frac{3}{2}$ [A. Stefanov *et al.* (2010)].
- For sufficiently *large* initial data, the local solutions break in a finite time [Y. Liu *et al.* (2009,2010) for $p = 1, 2$].
- For sufficiently *small* initial data in $\dot{H}^2_{\rm per}$, the local solutions are continued globally [D.P.,A.Sakovich (2010) for $p = 2$].
- For sufficiently *small* initial data in $\dot{H}^3_{\rm per}$, the local solutions are continued globally [R. Grimshaw, D.P. (2014) for $p = 1$].
- For $p = 1$ and $p = 2$, the reduced Ostrovsky equation is reduced to an integrable equation of the Klein–Gordon type.

Consider **travelling $2T$-periodic waves** $u(x,t) = U(x - ct)$ in the generalized reduced Ostrovsky equation:

$$(u_t + u^p u_x)_x = u, \quad p \in \mathbb{N}.$$

The wave profile satisfies the second-order ODE

$$\frac{d}{dz}\left[(c - U^p)\frac{dU}{dz}\right] + U(z) = 0, \quad U(-T) = U(T), \quad U'(-T) = U'(T),$$

where $z = x - ct$ and c is the wave speed.

After two integrations, the ODE is the Euler–Lagrange equation of the energy function $F(u) = H(u) + cQ(u)$ in $\dot{L}^2_{\rm per} \cap L^{p+2}$, where

$$H(u) = -\frac{1}{2}\|\partial_x^{-1} u\|^2_{L^2_{\rm per}} - \frac{1}{(p+1)(p+2)} \int_{-T}^{T} u^{p+2} dx,$$

and

$$Q(u) = \frac{1}{2}\|u\|^2_{L^2_{\rm per}}$$

are conserved energy and momentum of the reduced Ostrovsky equation.

Spectral stability

Traveling periodic wave U is a critical point of $F(u) = H(u) + cQ(u)$ in $\dot{L}^2_{\rm per} \cap L^{p+2}$. The Hessian operator is

$$L = P_0 \left(\partial_z^{-2} + c - U(z)^p \right) P_0 : \; \dot{L}^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T),$$

where $P_0 : L^2_{\rm per} \to \dot{L}^2_{\rm per}$ is the mean-zero projection operator.

Definition

We say that the traveling wave is **spectrally stable** if $\partial_z L : \dot{H}^1_{\rm per} \to \dot{L}^2_{\rm per}$ has no eigenvalues λ with $\mathrm{Re}(\lambda) > 0$.

Approaches to stability of traveling periodic waves:

- Orbital stability in $\dot{H}^3_{\rm per}$ (for $p = 1$) and $\dot{H}^2_{\rm per}$ (for $p = 2$) by using higher-order energy [E.R.Johnson, D.P. (2016)]
- Spectral stability in $\dot{L}^2_{\rm per}$ (for $p = 1$ and $p = 2$) from eigenvalues of $M\psi = \lambda \partial_z \psi$ in $L^2_{\rm per}$ [S. Hakkaev, *et al.* (2017)].
- Spectral stability in $\dot{L}^2_{\rm per}$ for any $p \in \mathbb{N}$ [A. Geyer, D.P. (2017)].

Orbital stability of periodic waves for $p = 1$

J. Brunelli & S. Sakovich (2013) found bi-infinite sequence of conserved quantities for the reduced Ostrovsky equation $(u_t + uu_x)_x = u$:

$$\begin{aligned}
& \dots \\
E_{-1} &= \int \left(\frac{1}{3} u^3 + (\partial_x^{-1} u)^2 \right) dx = -2H, \\
E_0 &= \int u^2 dx = 2Q \\
E_1 &= \int (1 - 3u_{xx})^{1/3} dx, \\
E_2 &= \int \frac{(u_{xxx})^2}{(1 - 3u_{xx})^{7/3}} dx \\
& \dots
\end{aligned}$$

Theorem (R.Grimshaw & D.P., 2014)

Let $u_0 \in H^3$ such that $1 - 3u_0''(x) > 0$ for all x. There exists a unique solution $u \in C(\mathbb{R}, H^3)$ to the reduced Ostrovsky equation with $u(0) = u_0$.

Traveling periodic wave U is a critical point of $F(u) = H(u) + cQ(u)$ in $\dot{L}^2_{\rm per} \cap L^3$ with

$$L_c = F''(U) = P_0 \left(\partial_z^{-2} + c - U(z)\right) P_0 : \; \dot{L}^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T),$$

where $P_0 : L^2_{\rm per} \to \dot{L}^2_{\rm per}$ is the mean-zero projection operator.

Let us normalize the period T to 2π. Then, $U = 0$ at $c = 1$, and

$$L_{c=1} = P_0(1 + \partial_z^{-2}) P_0 \quad \sigma(L_{c=1}) = \{1 - n^{-2}, \quad n \geq 1\},$$

where the spectrum is defined in $\dot{L}^2_{\rm per}(0,2\pi)$. All eigenvalues are positive except for the double zero eigenvalue.

Variational characterizations of periodic waves

Traveling periodic wave U is a critical point of $F(u) = H(u) + cQ(u)$ in $\dot{L}^2_{\rm per} \cap L^3$ with

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For the subharmonic perturbations in $\dot{L}^2_{\rm per}(0, 2\pi N)$ with $N \geq 1$, the spectrum is

$$\sigma(L_{c=1}) = \{1 - n^{-2} N^2, \quad n \geq 1\}.$$

There are $N - 1$ double negative eigenvalues and a double zero eigenvalue.

U is not a minimizer of $F(u) = H(u) + cQ(u)$.

Alternative variational characterizations of periodic waves

Traveling periodic wave U is also a critical point of

$$G(u) = R(u) - \frac{1}{(c^3 - 6I_c)^{2/3}} Q(u) \quad \text{in} \quad \dot{H}^3_{\mathrm{per}}$$

where

$$R(u) = -\int (1 - 3u_{xx})^{1/3}\, dx$$

and

$$I_c = \frac{1}{2}(c - U)^2 \left(\frac{dU}{dz}\right)^2 + \frac{c}{2}U^2 - \frac{1}{3}U^3 = \text{const in } z.$$

Here $c^3 - 6I_c > 0$, $U(z) < c$, and $1 - 3U''(z) > 0$ for smooth periodic waves.

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Here $c^3 - 6I_c > 0$, $U(z) < c$, and $1 - 3U''(z) > 0$ for smooth periodic waves.

The Hessian operator is

$$M_c = G''(U) = P_0 \left(\partial_z^2 (1 - 3U'')^{-5/3} \partial_z^2 - (c^3 - 6I_c)^{-2/3} \right) P_0 : \; \dot{H}^4_{\rm per} \to \dot{L}^2_{\rm per},$$

For $U = 0$ at $c = 1$ and for the subharmonic perturbations in $\dot{L}^2_{\rm per}(0, 2\pi N)$

$$M_{c=1} = P_0(-1 + \partial_z^4) P_0 \quad \sigma(M_{c=1}) = \{-1 + n^4 N^{-4}, \quad n \geq 1\}.$$

There are $N - 1$ double negative eigenvalues and a double zero eigenvalue.
U **is not a minimizer of** $G(u)$**.**

Mixed variational structure

Following

- N. Bottman, B. Deconinck, DCDS A (2009)
- B. Deconinck, T. Kapitula, Physics Letters A (2010)
- M. Nivala, B. Deconinck, Physica D (2010)
- N. Bottman, B. Deconinck, M. Nivala, J. Phys. A (2011)
- Th. Gallay, D.P., J. Diff. Eq. (2015)

we define a mixed variational structure for periodic waves U:

$$W_b(u) := G(u) - bF(u), \quad b \in \mathbb{R}.$$

Theorem (E.Johnson, D.P., 2016)

For sufficiently small $|c-1|$, U is a local nondegenerate (up to translational symmetry) minimizer of $W_b(u)$ in $\dot{H}^3_{\rm per}(0, 2\pi N)$ for every $b \in (b_-, b_+)$, where $b_\pm$ are given asymptotically by

$$b_\pm = \frac{1}{2} \pm \frac{3}{\sqrt{2}}\sqrt{c-1} + \mathcal{O}(c-1), \quad \text{as} \quad c \to 1.$$

Numerical results: periodic wave U

Galerkin-Fourier approximation

$$U(z) = \sum_{n=1}^{N} A_n \cos(nz),$$

where $a = |A_1|$ is taken as the wave amplitude (depends on $c > 1$).

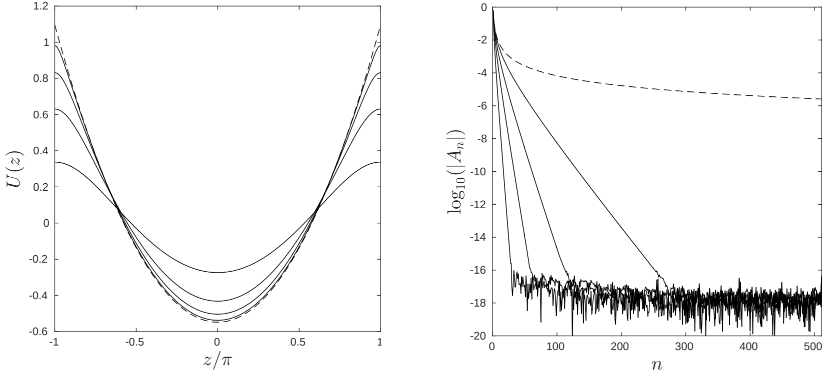

Figure: (a) The 2π-periodic solutions of the reduced Ostrovsky equation. (b) The Fourier coefficients of the trigonometric approximation.

Numerical results: U as a minimizer of W

The mixed variational structure yields

$$W_b(u) := G(u) - bF(u),$$

and U is a critical point of W for every $b \in \mathbb{R}$.

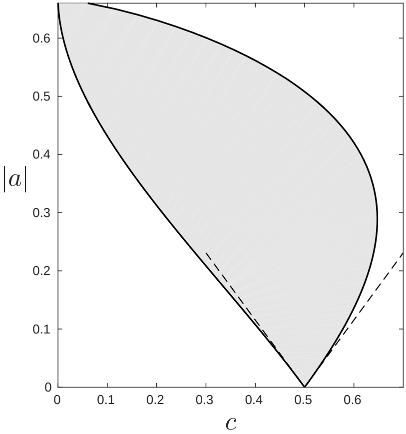

Figure: The region of the (b, a) plane where U is a minimizer of $W_b(u)$.

Orbital stability of periodic waves for $p = 2$

J. Brunelli (2005) found bi-infinite sequence of conserved quantities for the modified reduced Ostrovsky equation $(u_t + u^2 u_x)_x = u$:

$$\begin{aligned}
&\dots \\
E_{-1} &= \int \left(\frac{1}{12} u^4 + (\partial_x^{-1} u)^2 \right) dx = -2H, \\
E_0 &= \int u^2 dx = 2Q, \\
E_1 &= \int (1 - u_x^2)^{1/2} dx, \\
E_2 &= \int \frac{u_{xx}^2}{(1 - u_x^2)^{5/2}} dx, \\
&\dots
\end{aligned}$$

Theorem (D.P. & A. Sakovich, 2010)

Let $u_0 \in H^2$ such that $\|u_0'\|_{L^2}^2 + \|u_0''\|_{L^2}^2 < 1$. There exists a unique solution $u \in C(\mathbb{R}, H^2)$ to the modified reduced Ostrovsky equation with $u(0) = u_0$.

Two variational characterizations of periodic waves

Traveling periodic wave U is a critical point of $F(u) = H(u) + cQ(u)$ in $\dot{L}^2_{\rm per} \cap L^4$ with

$$L_c = F''(U) = P_0 \left(\partial_z^{-2} + c - U(z)^2 \right) P_0 : \; \dot{L}^2_{\rm per} \to \dot{L}^2_{\rm per},$$

where $P_0 : L^2_{\rm per} \to \dot{L}^2_{\rm per}$ is the mean-zero projection operator.

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Traveling periodic wave U is also a critical point of

$$G(u) = R(u) - \frac{1}{2(c^2 - 2I_c)^{1/2}} Q(u) \quad \text{in} \;\; \dot{H}^2_{\rm per}$$

where

$$R(u) = -\int \left(1 - u_x^2\right)^{1/2} dx$$

and

$$I_c = \frac{1}{2}(c - U^2)^2 \left(\frac{dU}{dz} \right)^2 + \frac{c}{2} U^2 - \frac{1}{2} U^4 = \text{const in } z.$$

Here $c^2 - 2I_c > 0$, $U(z)^2 < c$, and $|U'(z)| < 1$ for smooth periodic waves.

U is not a minimizer of neither $F(u)$ nor $G(u)$ in $\dot{L}^2_{\rm per}(0, 2\pi N)$.

Mixed variational structure

Let us define now the mixed variational structure for periodic waves U:

$$W_b(u) := G(u) - bF(u), \quad b \in \mathbb{R}.$$

U is a critical point of W.

Theorem (E.Johnson, D.P., 2016)

For sufficiently small $|c-1|$, U is a local nondegenerate (up to translational symmetry) minimizer of $W_b(u)$ in $\dot{H}^2_{\rm per}(0,2\pi N)$ for every $b \in (b_-, b_+)$, where $b_\pm$ are given asymptotically by

$$b_\pm = 2 \pm 4\sqrt{2}\sqrt{c-1} + \mathcal{O}(c-1), \quad \text{as} \quad c \to 1.$$

The mixed variational structure yields

$$W_b(u) := G(u) - bF(u),$$

and U is a critical point of W for every $b \in \mathbb{R}$.

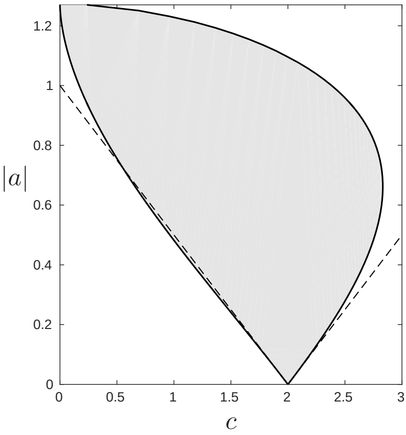

Figure: The region of the (b, a) plane where U is a minimizer of $W_b(u)$.

The **travelling $2T$-periodic waves** $u(x,t) = U(x-ct)$ satisfies the second-order ODE

$$\frac{d}{dz}\left[(c-U^p)\frac{dU}{dz}\right] + U(z) = 0, \quad U(-T) = U(T), \quad U'(-T) = U'(T),$$

with the first-order invariant

$$E = \frac{1}{2}(c-U^p)^2\left(\frac{dU}{dz}\right)^2 + \frac{c}{2}U^2 - \frac{1}{p+2}U^{p+2} = \text{const},$$

where $z = x - ct$ and c is the wave speed.

Theorem (A.Geyer, D.P., 2017)

For every $c > 0$ and $p \in \mathbb{N}$, there exists a smooth family of periodic solutions $U \in \dot{L}^2_{\text{per}}(-T,T) \cap H^\infty_{\text{per}}(-T,T)$ parameterized by $E \in (0, E_c)$ such that the energy-to-period map $E \mapsto 2T$ is strictly monotonically decreasing.

Existence theorem on the phase plane

The first-order invariant

$$E = \frac{1}{2}(c - U^p)^2 \left(\frac{dU}{dz}\right)^2 + \frac{c}{2}U^2 - \frac{1}{p+2}U^{p+2} = \text{const}$$

yield integral curves on the (U, U') phase plane.

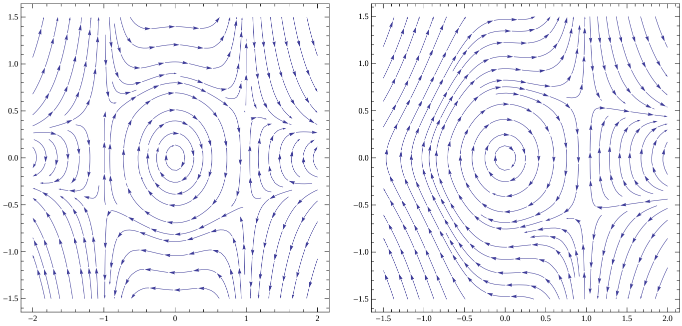

Figure: Phase portraits for $p = 2$ (left) and $p = 1$ (right).

Monotonicity of the map $E \mapsto 2T$

It follows from

$$E = \frac{1}{2}(c - U^p)^2 \left(\frac{dU}{dz}\right)^2 + \frac{c}{2}U^2 - \frac{1}{p+2}U^{p+2} = \mathrm{const}$$

that

$$2T(E) = \int_{\gamma_E} \frac{du}{v} = 2\int_{u_-(E)}^{u_+(E)} \frac{\sqrt{B(u)}du}{\sqrt{E - A(u)}},$$

where $A(u) = \frac{c}{2}u^2 - \frac{1}{p+2}u^{p+2}$ and $B(u) = \frac{1}{2}(c - u^p)^2$.

- The integrand is singular at the turning points $u_\pm(E)$ where $A(u_\pm) = E$.
- Derivative in E can not be applied separately to the integrand and the limits of integration.

Monotonicity of the map $E \mapsto 2T$

Following

- M. Frau, F. Manosas, J. Villadelprat, Transactions AMS (2011)
- A. Farijo, J. Villadelprat, J. Diff. Eq. (2014)

one can rewrite it

$$\begin{aligned} 2ET(E) &= \int_{\gamma_E} B(u)v\,du + \int_{\gamma_E} A(u)\frac{du}{v} \\ &= \int_{\gamma_E} \left[B(u) + \left(\frac{2A(u)B(u)}{A'(u)} \right)' - \frac{A(u)B'(u)}{A'(u)} \right] v\,du, \end{aligned}$$

where the integrand is now free of singularities at the turning points.

Then, applying derivative in E, we obtain

$$2T(E) + 2E\,T'(E) = \int_{\gamma_E} \frac{B(u) + G(u)}{2B(u)v}du$$

and the final expression

$$T'(E) = -\frac{p}{4(2+p)E} \int_{\gamma_E} \frac{u^p}{(c-u^p)} \frac{du}{v} < 0.$$

Existence theorem on the parameter plane

For fixed c, the map $E \mapsto 2T$ is monotonically decreasing for $E \in (0, E_c)$ with $T(0) = \pi c^{1/2}$ and $T(E_c) = T_1 c^{1/2}$, where $T_1 < \pi$ is independent of c.

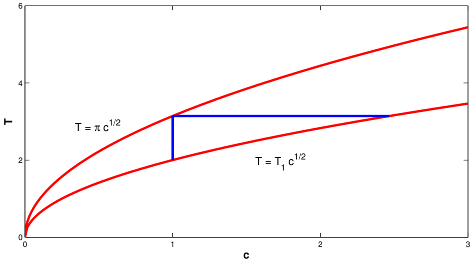

Figure: The existence region for smooth periodic waves in the (T, c)-parameter plane.

For fixed T, the map $c \mapsto E$ is monotonically increasing for $c \in (T^2\pi^{-2}, T^2 T_1^{-2})$.

Spectral stability in the generalized reduced Ostrovsky equation

The $2T$**-periodic wave** U is a critical point of $F(u) = H(u) + cQ(u)$, where

$$H(u) = -\frac{1}{2}\|\partial_x^{-1} u\|^2_{L^2_{\rm per}} - \frac{1}{(p+1)(p+2)} \int_{-T}^{T} u^{p+2} dx,$$

$$Q(u) = \frac{1}{2}\|u\|^2_{L^2_{\rm per}}$$

The Hessian operator is

$$L = P_0 \left(\partial_z^{-2} + c - U(z)^p \right) P_0 : \; \dot{L}^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T),$$

where $P_0 : L^2_{\rm per} \to \dot{L}^2_{\rm per}$ is the zero-mean projection operator.

Theorem (A.Geyer, D.P., 2017)

For every $c > 0$, $p \in \mathbb{N}$, and U, the operator L in $\dot{L}^2_{\rm per}(-T,T)$ has a simple negative eigenvalue, a simple zero eigenvalue associated with $\mathrm{Ker}(L) = \mathrm{span}\{\partial_z U\}$, and the rest of the spectrum is strictly positive. Moreover, the operator L is positive under the fixed-momentum constraint:

$$L^2_c = \left\{ u \in \dot{L}^2_{\rm per}(-T,T) : \quad \langle U, u \rangle_{L^2_{\rm per}} = 0 \right\}.$$

An argument about the spectrum of L

Fix $T > 0$ and consider the Hessian operator

$$L = P_0 \left(\partial_z^{-2} + c - U(z)^p \right) P_0 : \ \dot{L}^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T).$$

At $c = T^2\pi^{-2}$, we have $U = 0$ and

$$L_0 = P_0(c + \partial_z^{-2})P_0 \quad \sigma(L_0) = \{c(1 - n^{-2}), \quad n \geq 1\}.$$

All eigenvalues are positive except for the double zero eigenvalue. For $c > T^2\pi^{-2}$, L_0 has only simple zero eigenvalue and a simple negative eigenvalue.

Lemma

The zero eigenvalue of L is simple if $T'(E) \neq 0$.

The family of operators L is iso-spectral with respect to parameter c.

An argument about the constraint L^2_c

Fix $T > 0$ and consider the Hessian operator

$$L = P_0 \left(\partial_z^{-2} + c - U(z)^p \right) P_0 : \; \dot{L}^2_{\text{per}}(-T,T) \to \dot{L}^2_{\text{per}}(-T,T).$$

under the scalar constraint

$$L^2_c = \left\{ u \in \dot{L}^2_{\text{per}}(-T,T) : \quad \langle U, u \rangle_{L^2_{\text{per}}} = 0 \right\}.$$

The operator L is positive under the constraint if

$$\langle L^{-1} U, U \rangle_{L^2_{\text{per}}} < 0,$$

where $U \perp \text{Ker}(L) = \text{span}(\partial_z U)$.

For fixed $T > 0$, $L \partial_c U = -U$ yields $\partial_c U = -L^{-1} U \in \dot{L}^2_{\text{per}}(-T,T)$, so that

$$\langle L^{-1} U, U \rangle_{L^2_{\text{per}}} = -\frac{1}{2} \frac{d}{dc} \|U\|^2_{L^2_{\text{per}}} < 0,$$

the latter inequality can be proved for every $p > 0$ and for every $c > 0$.

Summary

For the generalized reduced Ostrovsky equation with an integer p,

$$(u_t + u^p u_x)_x = u,$$

we have shown two stability results for the travelling periodic waves:

- Minimization property for higher-order energy in $\dot{H}^s_{\rm per}$-spaces for $p = 1$ and $p = 2$
- Spectral stability in $\dot{L}^2_{\rm per}$ for any $p \in \mathbb{N}$

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- Spectral stability in $\dot{L}^2_{\text{per}}$ for any $p \in \mathbb{N}$

Spectral stability for $p \geq 3$ cannot be transferred to the orbital stability results because the global well-posedness is not available in $\dot{L}^2_{\text{per}} \cap L^{p+2}$, where the energy and momentum functions $H(u)$ and $Q(u)$ are defined.