Smooth and peaked waves in the reduced Ostrovsky equation

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Ostrovsky equation in a physical context

The *Korteweg–De Vries equation* (1895) governs dynamics of small-amplitude long waves in a fluid:

$$u_t + uu_x + \beta u_{xxx} = 0,$$

where u is a real-valued function of (x,t). It arises from expansion of the dispersion relation for linear waves $e^{i(kx-\omega t)}$:

$$\omega^2 = c^2k^2 + \beta k^4 + \mathcal{O}(k^6) \quad \Rightarrow \quad \omega - ck = \frac{1}{2c}\beta k^3 + \mathcal{O}(k^5).$$

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The *Kadomtsev–Petviasvhili equation* (1970) models diffraction:

$$(u_t + uu_x + \beta u_{xxx})_x + u_{yy} = 0,$$

as follows from:

$$\omega^2 = c^2(k^2+p^2) + \beta(k^2+p^2)^2 + \cdots \quad \Rightarrow \quad \omega - ck = \frac{\beta}{2c}k^3 + \frac{p^2}{2ck} + \cdots$$

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The *Ostrovsky equation* (1978) models rotation:

$$(u_t + uu_x + \beta u_{xxx})_x = \gamma^2 u,$$

as follows from:

$$\omega^2 = \gamma^2 + c^2k^2 + \beta k^4 + \cdots \quad \Rightarrow \quad \omega - ck = \frac{\beta}{2c}k^3 + \frac{\gamma^2}{2ck} + \cdots$$

The reduced Ostrovsky equation

As $\beta \to 0$, we obtain *the reduced Ostrovsky equation*

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For internal or interfacial waves, *the reduced modified Ostrovsky equation* is more relevant [Grimshaw, 1985]:

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$$(u_t + u^2 u_x)_x = u.$$

Note the difference from *the short-pulse equation* derived as a model for propagation of pulses with few cycles [Schäfer, Wayne 2004]:

$$(u_t - u^2 u_x)_x = u.$$

Plan of my talk

Consider *the generalized reduced Ostrovsky equation*

$$(u_t + u^p u_x)_x = u, \quad p \in \mathbb{N}.$$

- Cauchy problem in Sobolev spaces:
  - Local solutions with zero mass constraint
  - Global smooth solutions
  - Wave breaking in a finite time
- Existence of periodic traveling waves:
  - A family of smooth periodic waves
  - A peaked periodic wave at the terminal point
  - No cusped periodic waves
- Stability of periodic traveling waves:
  - Spectral stability of smooth waves
  - Spectral and linear instability of peaked waves

Cauchy problem in Sobolev spaces

Consider Cauchy problem for *the reduced Ostrovsky equation*

$$\begin{cases} (u_t + u^p u_x)_x = u, \\ u|_{t=0} = u_0. \end{cases}$$

- Local well-posedness for $u_0 \in H^s$ with $s > 3/2$
  [Stefanov et. al., 2010]
- Zero mass constraint is necessary in the periodic domain: $\int_{-\pi}^{\pi} u_0(x) dx = 0$.

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- Zero mass constraint is necessary in the periodic domain: $\int_{-\pi}^{\pi} u_0(x) dx = 0$.
- Local solutions break in finite time for large initial data.
  [Liu & P. & Sakovich 2009, 2010 for $p = 1$, $p = 2$]
- Global solutions exist for small initial data.
  [Grimshaw & P. 2014 for $p = 1$]

Global solutions for small initial data

Theorem (Grimshaw & P., 2014)

Let $u_0 \in H^3$ such that $1 - 3u_0''(x) > 0$ for all x. There exists a unique solution $u(t) \in C(\mathbb{R}, H^3)$ with $u(0) = u_0$.

This result is based on the preliminary works:

- Hone & Wang (2003) obtained Lax pair

$$\left\{ \begin{array}{c} 3\lambda\psi_{xxx} + (1 - 3u_{xx})\psi = 0, \\ \psi_t + \lambda\psi_{xx} + u\psi_x - u_x\psi = 0, \end{array} \right.$$

- Kraenkel, LeBlond, & Manna (2014) showed equivalence to the Bullough–Dodd (Tzitzeica) equation

$$\frac{\partial^2 V}{\partial t \partial z} = e^{-2V} - e^{V}.$$

Conserved quantities for the reduced Ostrovsky equation

Brunelli & Sakovich (2013) found bi-infinite sequence of conserved quantities for the reduced Ostrovsky equation:

$$
\begin{aligned}
& \ldots \\
E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{3} u^3 + (\partial_x^{-1} u)^2 \right) dx, \\
E_0 &= \int_{\mathbb{R}} u^2 dx \\
E_1 &= \int_{\mathbb{R}} \left[(1 - 3u_{xx})^{1/3} - 1 \right] dx, \\
E_2 &= \int_{\mathbb{R}} \frac{(u_{xxx})^2}{(1 - 3u_{xx})^{7/3}} dx \\
& \ldots
\end{aligned}
$$

Characteristic variable for the reduced Ostrovsky equation

Start with local solutions $u \in C([0,T], H^3)$ to

$$(u_t + uu_x)_x = u, \quad x \in \mathbb{R}, \quad t \in [0,T].$$

Let $x = x(\xi, t)$ satisfy $x = \xi + \int_0^t U(\xi, t')dt'$ with $u(x,t) = U(\xi, t)$.
The transformation $\xi \to x$ is invertible if

$$\phi(\xi, t) := \frac{\partial x}{\partial \xi} = 1 + \int_0^t U_\xi(\xi, t')dt' \neq 0.$$

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Let us introduce $f(x,t) = (1 - 3u_{xx})^{1/3} = F(\xi,t)$. Then,

$$F(\xi,t)\phi(\xi,t) = F_0(\xi)$$

and

$$\frac{\partial^2}{\partial t \partial \xi} \log(F) = \frac{1}{3} F_0(\xi)(F^2 - F^{-1}).$$

Towards global existence

- If $1 - 3u_0''(x) > 0$ for all $x \in \mathbb{R}$, then $F_0(x) > 0$. Setting

$$z := -\frac{1}{3}\int_0^{\xi} F_0(\xi')d\xi', \quad F(\xi,t) := e^{-V(z,t)},$$

yields the Tzitzéica equation

$$\frac{\partial^2 V}{\partial t \partial z} = e^{-2V} - e^{V}.$$

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- A local solution $V \in C([0,T], H^1(\mathbb{R}))$ to the Tzitzéica equation follows from a local solution $u \in C([0,T], H^3(\mathbb{R}))$:

$$V(z,t) = -\frac{1}{3}\log\left(1 - 3u_{xx}(x,t)\right).$$

Towards global existence

- The H^1 norm of $V \in C([0,T], H^1(\mathbb{R}))$ is bounded by the conserved quantities

$$Q_1 = \int_{\mathbb{R}} \left(2e^V + e^{-2V} - 3\right) dz, \quad Q_2 = \int_{\mathbb{R}} \left(\frac{\partial V}{\partial z}\right)^2 dz.$$

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- Together with the invertible coordinate transformation

$$u_{xx}(x,t) = \frac{1}{3}\left(1 - e^{-3V(z,t)}\right)$$

and conserved quantities

$$E_0 = \int_{\mathbb{R}} u^2 dx, \quad E_2 = \int_{\mathbb{R}} \frac{(u_{xxx})^2}{(1-3u_{xx})^{7/3}} dx$$

this controls the H^3 norm of $u \in C([0,T], H^3(\mathbb{R}))$.

Wave breaking for large initial data

Lemma

Let $u_0 \in H^2_{\rm per}$*. The local solution* $u \in C([0,T), H^2_{\rm per})$ *blows up in a finite time* $T < \infty$ *in the sense* $\lim_{t\uparrow T} \|u(\cdot,t)\|_{H^2} = \infty$ *if and only if*

$$\lim_{t\uparrow T} \inf_x u_x(t,x) = -\infty, \quad \textit{while} \quad \lim_{t\uparrow T} \sup_x |u(t,x)| < \infty.$$

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Theorem (Hunter, 1990)

Let $u_0 \in C^1_{\rm per}$ *and define*

$$\inf_{x\in\mathbb{S}} u_0'(x) = -m \quad \text{and} \quad \sup_{x\in\mathbb{S}} |u_0(x)| = M.$$

If $m^3 > 4M(4+m)$, *a smooth solution* $u(t,x)$ *breaks in a finite time.*

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Theorem (Liu, P. & Sakovich, 2010)

Assume that $u_0 \in H^2_{\rm per}$. The solution breaks if

$$\text{either} \quad \int_{\mathbb{S}} (u_0'(x))^3 \, dx < -\left(\frac{3}{2}\|u_0\|_{L^2}\right)^{3/2}, \tag{1}$$

$$\text{or} \quad \exists x_0 : \quad u_0'(x_0) < -1 \left(\|u_0\|_{L^\infty} + T_1 \|u_0\|_{L^2}\right)^{\frac{1}{2}}. \tag{2}$$

Proof of the sufficient condition (1)

Direct computation gives

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= -2\int_{\mathbb{S}} u_x^4\,dx + 3\int_{\mathbb{S}} u u_x^2\,dx \\ &\leq -2\|u_x\|_{L^4}^4 + 3\|u\|_{L^2}\|u_x\|_{L^4}^2. \end{aligned}$$

By Hölder's inequality, we have

$$|V(t)| \leq \|u_x\|_{L^3}^3 \leq \|u_x\|_{L^4}^3, \quad V(t) = \int_{\mathbb{S}} u_x^3(t,x)\,dx < 0.$$

Let $Q_0 = \|u\|_{L^2}^2 = \|u_0\|_{L^2}^2$ and $V(0) < -\left(\frac{3}{2}Q_0\right)^{\frac{3}{2}}$. Then,

$$\frac{dV}{dt} \leq -2\left(|V|^{\frac{2}{3}} - \frac{3Q_0}{4}\right)^2 + \frac{9Q_0^2}{8},$$

There is $T < \infty$ such that $V(t) \to -\infty$ as $t \uparrow T$.

Proof of the sufficient condition (2)

Introduce characteristic variables for $u_t + uu_x = \partial_x^{-1} u$:

$$x = X(\xi, t), \quad u(x,t) = U(\xi, t), \quad \partial_x^{-1} u(x,t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\left\{ \begin{array}{l} \dot{X}(t) = U, \\ X(0) = \xi, \end{array} \right. \qquad \left\{ \begin{array}{l} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{array} \right.$$

Let $V(\xi, t) = u_x(t, X(\xi, t))$. Then

$$\dot{V} = -V^2 + U \quad \Rightarrow \quad \dot{V} \leq -V^2 + (\|u_0\|_{L^\infty} + t\|u_0\|_{L^2})$$

There is $T < \infty$ such that $V(t) \to -\infty$ as $t \uparrow T$.

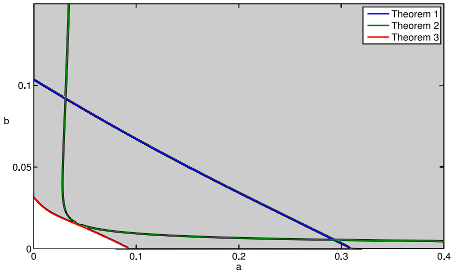
Numerical simulations

By using a pseudospectral method based on Fourier series:

$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k - \frac{ik}{2}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^2\right]_k, \quad k \neq 0, \quad t > 0,$$

where the initial condition is

$$u_0(x) = a\cos(x) + b\sin(2x),$$



Evolution of the cosine initial data

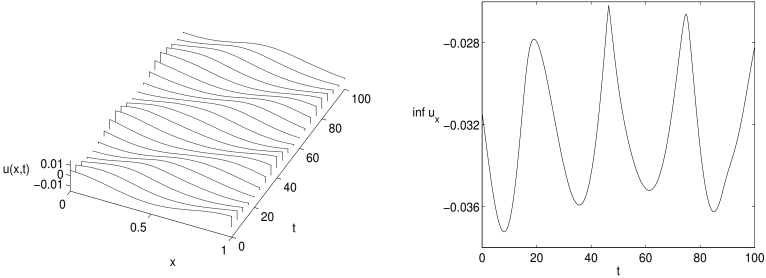


Figure: Solution surface $u(t,x)$ (left) and $\inf_{x\in\mathbb{S}} u_x(t,x)$ versus t (right) for $a = 0.005$, $b = 0$.

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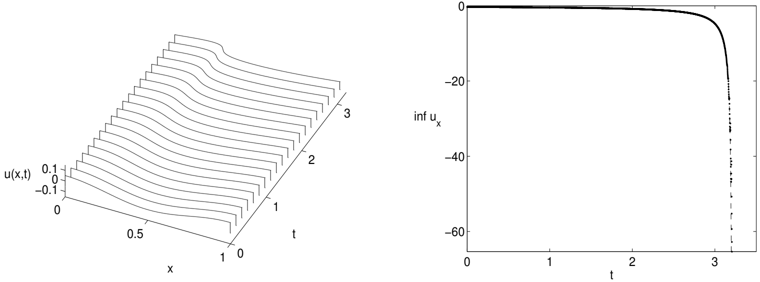

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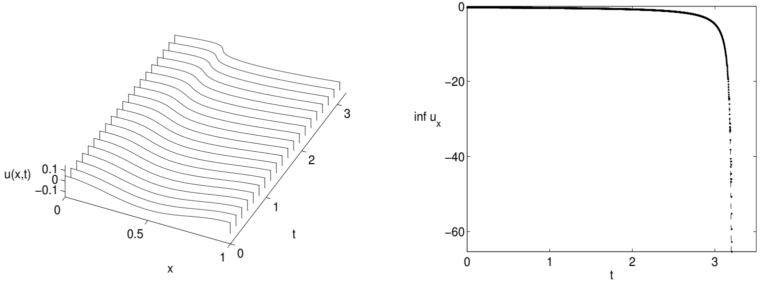

Figure: Solution surface $u(t,x)$ (left) and $\inf_{x\in\mathbb{S}} u_x(t,x)$ versus t (right) for $a = 0.05$, $b = 0$.

Conjecture: The smooth solution breaks in a finite time if $u_0 \in H^3$ yields sign-indefinite $1 - 3u_0''(x)$.

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Smooth traveling wave solutions

Traveling wave solutions are solutions of the form

$$u(x,t) = U(x - ct),$$

where $z = x - ct$ is the travelling wave coordinate and c is the wave speed. The wave profile U is $2T$-periodic for fixed c.

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The wave profile U satisfies the boundary-value problem

$$\frac{d}{dz}\left((c - U^p)\frac{dU}{dz}\right) + U(z) = 0, \qquad \left.\begin{array}{l} U(-T) = U(T), \\ U'(-T) = U'(T), \end{array}\right\} \quad \text{(ODE)}$$

where $\int_{-T}^{T} U(z)dz = 0$, i.e. the periodic waves have zero mean.

ODE technique

Let $c > 0$ and $p \in \mathbb{N}$. A function U is a smooth periodic solution of

$$\frac{d}{dz}\left((c - U^p)\frac{dU}{dz}\right) + U = 0 \tag{ODE}$$

iff $(u, v) = (U, U')$ is a periodic orbit γ_E of the planar system

$$\begin{cases} u' = v, \\ v' = \dfrac{-u + pu^{p-1}v^2}{c - u^p}, \end{cases}$$

which has the first integral

$$E(u, v) = \frac{1}{2}(c - u^p)^2 v^2 + \frac{c}{2}u^2 - \frac{1}{p+2}u^{p+2}.$$

The periodic wave U is smooth iff $c - U(z)^p > 0$ for every z.

Existence of smooth periodic traveling waves

Let $c > 0$ and $p \in \mathbb{N}$. The first integral is

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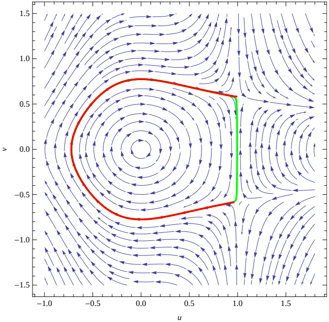

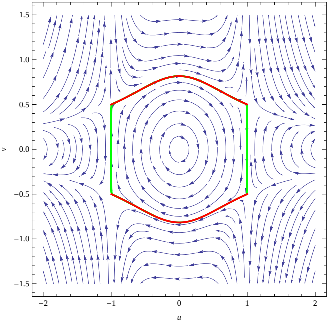

There exists a smooth family of periodic solutions parametrized by the energy $E \in (0, E_c)$, where $2T$ depends on E.

Properties of smooth periodic waves

Theorem (Geyer & P., 2017)

For fixed c, the map $E \mapsto T$ is decreasing with $T(0) = \pi c^{1/2}$.
For fixed T, the map $E \mapsto c$ is increasing with $c(0) = T^2/\pi^2$.

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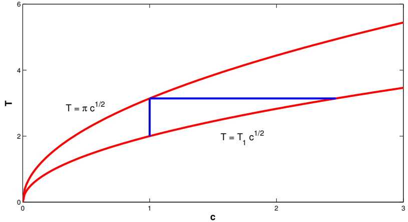
The map $E \mapsto T$ for fixed c is transferred to the map $E \mapsto c$ for fixed T by the scaling transformation

$$U(z;c) = c^{1/p}\tilde{U}(\tilde{z}), \quad z = c^{1/2}\tilde{z}, \quad T = c^{1/2}\tilde{T},$$

where $\tilde{U}$ is a $2\tilde{T}$-periodic solution of the same (ODE) with $c = 1$.



Peaked 2π-periodic wave for $p = 1$

The 2π periodic traveling wave solutions $U(z)$ satisfy the BVP

$$\left\{ \begin{array}{l} [c - U(z)]\, U'(z) + (\partial_z^{-1} U)(z) = 0, \quad z \in (-\pi, \pi) \\ U(-\pi) = U(\pi), \end{array} \right.$$

where $z = x - ct$ and $\int_{-\pi}^{\pi} U(z) dz = 0$.

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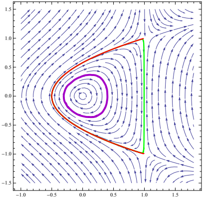
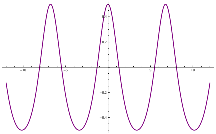
Theorem (Existence of smooth periodic waves)

There exists $c_* > 1$ such that for every $c \in (1, c_*)$, the BVP admits a unique smooth periodic wave U satisfying $U(z) < c$ for $z \in [-\pi, \pi]$.

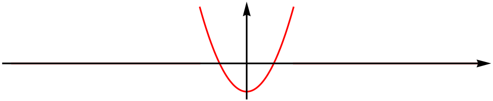




Peaked periodic wave for $p = 1$

For $c = c_* := \pi^2/9$ there exists a solution with parabolic profile

$$U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi],$$



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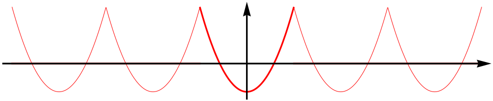

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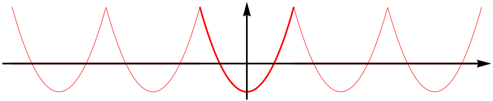

The peaked periodic wave $U_* \in H^s_{\mathrm{per}}(-\pi, \pi)$ for $s < 3/2$:

$$U_*(z) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{3n^2} \cos(nz),$$

with $U_*(\pm\pi) = c_*$ and $U_*'(\pm\pi) = \pm\pi/3$.

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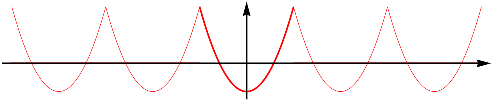

The peaked wave satisfies the border case: $1 - 3U_*''(z) = 0$ for $z \in (-\pi, \pi)$.

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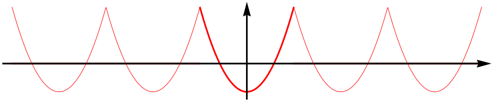
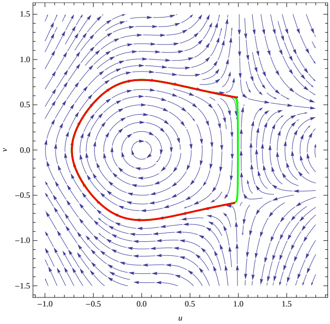


Theorem (Geyer & P, 2019)

The peaked periodic wave U_* is the unique *peaked solution with the jump at $z = \pm\pi$.*

Other peaked periodic traveling waves ?



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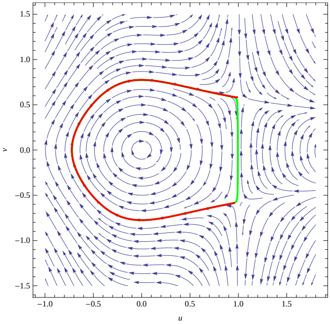

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A more general proof was given for $u_t + uu_x = \partial_x^{-r} u$ with $r > 1$:

[Bruell & Dhara, 2019]

Plan of my talk

Consider *the generalized reduced Ostrovsky equation*

$$(u_t + u^p u_x)_x = u, \quad p \in \mathbb{N}.$$

- Cauchy problem in Sobolev spaces:
  - Local solutions with zero mass constraint
  - Global smooth solutions
  - Wave breaking in a finite time
- Existence of periodic traveling waves:
  - A family of smooth periodic waves
  - A peaked periodic wave at the terminal point
  - No cusped periodic waves
- Stability of periodic traveling waves:
  - Spectral stability of smooth waves
  - Spectral and linear instability of peaked waves

Summary of stability results

The generalized reduced Ostrovsky equation

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where $p \in \mathbb{N}$.

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- $p = 1, 2$: Linear and spectral instability of the limiting peaked wave [Geyer & P., 2019]

Broader picture on stability of peaked periodic waves

- KdV equation: smooth solutions are stable, no peaked solutions
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- Ostrovsky equation: all smooth solutions are stable, but the limiting *peaked solution is unstable*.
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Using $u(t,x) = U(z) + v(z)e^{\lambda t}$, where $z = x - ct$, the spectral stability problem for a perturbation of the wave profile U is given by

$$\boxed{\partial_z L v = \lambda v}$$

with the self-adjoint linear operator

$$L = P_0 \left(\partial_z^{-2} + c - U^p\right) P_0 : \dot{L}^2_{\rm per}(-T,T) \to \dot{L}^2_{\rm per}(-T,T),$$

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Definition

The travelling wave is *spectrally stable* with respect to co-periodic perturbations if the spectral problem $\partial_z L v = \lambda v$ with $v \in H^1_{\rm per}(-T,T)$ has no eigenvalues $\lambda \notin i\mathbb{R}$.

Spectral stability - course of action

- Construct an augmented Lyapunov functional:

$$F[u] := H[u] + cQ[u],$$

where

$$\text{(energy)} \quad H[u] = -\frac{1}{2}\|\partial_x^{-1}u\|^2_{L^2_{\text{per}}} - \frac{1}{(p+1)(p+2)}\int_{-T}^{T} u^{p+2}dx$$

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Theorem (Geyer & P., 2017)

a traveling wave U is a local constrained minimizer of the energy $H[u]$ with fixed momentum $Q[u]$.

Spectral stability - course of action

- The constraint of fixed momentum $Q[u] := \frac{1}{2}\|u\|^2_{L^2_{\rm per}} = q$ is equivalent to restricting the self-adjoint linear operator L to the subspace

$$U^{\perp} = \left\{ v \in \dot{L}^2_{\rm per}(-T,T) : \quad \langle U, v \rangle_{L^2_{\rm per}} = 0 \right\}$$

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Indeed,

$$\begin{aligned} 0 = Q[U+v] - Q[U] &= \frac{1}{2}\int_{-T}^{T} (U+v)^2 dz - \frac{1}{2}\int_{-T}^{T} U^2 dz \\ &= \int_{-T}^{T} U\, v\, dz + O(v^2) \\ &= \langle U, v \rangle. \end{aligned}$$

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- *Hamilton-Krein index theory* for the spectral problem

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states that [Haragus & Kapitula, 08]

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- **Result:** the smooth periodic wave U is stable.

Operator L restricted to constrained space

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This is true if the following two conditions hold:
[Vakhitov-Kolokolov, 1975], [Grillakis–Shatah–Strauss, 1987]

- L has exactly one negative eigenvalue,
  a simple zero eigenvalue with eigenvector $\partial_z U$,
  and the rest of its spectrum is positive and bounded away from 0
- $\langle L^{-1} U, U \rangle = -\frac{d}{dc} \|U\|^2_{L^2_{\rm per}(-T,T)} < 0$, where the period T is fixed.

Both conditions are proven using strict monotonicity of the energy-to-period map $T(E)$.

Spectral properties of the operator L

Recall the self-adjoint linear operator

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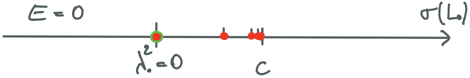
When $E \to 0$, then $U \to 0$, $T(E) \to T(0) = \sqrt{c}\pi$, and

$$L \to L_0 = P_0 \left(\partial_z^{-2} + c\right) P_0.$$

$\sigma(L_0) = \{c(1 - n^{-2}), n \in \mathbb{Z} \setminus \{0\}\}$ all eigenvalues are double.



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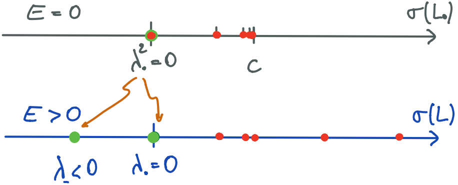

When $E > 0$ the double zero eigenvalue splits into a simple negative eigenvalue and a simple zero eigenvalue of L.

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Consider the eigenvalue problem

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Zero eigenvalue $\lambda_0 = 0$:

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If $T'(E) \neq 0$, then U_E is not $2T(E)$-periodic: $\mathrm{Ker}(L) = \mathrm{span}\{U_z\}$

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As a result, $L|_{U^\perp}$ is positive.

Spectral instability of the peaked periodic wave: $p = 1$

Let $u = U + v$ and consider the linearized evolution for a co-periodic perturbation v to the travelling wave U:

$$\left\{ \begin{array}{l} v_t + \partial_z \left[(U_*(z) - c_*) v \right] = \partial_z^{-1} v, \quad t > 0, \\ v|_{t=0} = v_0, \end{array} \right.$$

or equivalently

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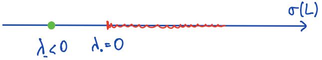
Lemma

The spectrum of the self-adjoint operator L is $\sigma(L) = \{\lambda_-\} \cup \left[0, \frac{\pi^2}{6}\right]$.



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The spectral stability problem can not be solved by applying standard energy methods due to the lack of coercivity.

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where $\dot{L}^2_{\rm per}$ is the L^2 space of periodic function with zero mean. Domain of $\partial_z L$ in $\dot{L}^2_{\rm per}$ is larger than $H^1_{\rm per}$:

$$\text{dom}(\partial_z L) = \left\{ v \in \dot{L}^2_{\rm per} : \quad \partial_z \left[(c_* - U_*) v \right] \in \dot{L}^2_{\rm per} \right\}.$$

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Definition

The travelling wave U is *linearly unstable* if there exists $v_0 \in \mathrm{dom}(\partial_z L)$ such that the unique global solution $v \in C(\mathbb{R}, \mathrm{dom}(\partial_z L))$ satisfies

$$\|v(t)\|_{L^2} \geq C e^{\lambda_0 t} \|v_0\|_{L^2}, \quad t > 0.$$

for some $\lambda_0 > 0$.

Linear instability of the peaked periodic wave

▷ **Step 1**: The *truncated problem*

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Method of characteristics. The characteristic curves $z = Z(s,t)$ are found explicitly and the solution of $V(s,t) := v(Z(s,t),t)$ is

$$V(s,t) = \frac{1}{\pi^2}[\pi \cosh(\pi t/6) - s \sinh(\pi t/6)]^2 v_0(s), \quad s \in [-\pi,\pi], \quad t \in \mathbb{R}.$$

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This yields the linear instability result for the truncated problem:

Lemma

For every $v_0 \in \mathrm{dom}(\partial_z L)$ $\exists!$ global solution $v \in C(\mathbb{R}, \mathrm{dom}(\partial_z L))$. If v_0 is odd, then the global solution satisfies

$$\frac{1}{2}\|v_0\|_{L^2} e^{\pi t/6} \le \|v(t)\|_{L^2} \le \|v_0\|_{L^2} e^{\pi t/6}, \quad t>0.$$

Linear instability of the peaked periodic wave

▷ **Step 2**: The *full evolution problem*

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Lemma

For every $v_0 \in \text{dom}(\partial_z L)$ $\exists!$ *global solution* $v \in C(\mathbb{R}, \text{dom}(\partial_z L))$. *If v_0 is odd, then the solution satisfies*

$$C\|v_0\|_{L^2} e^{\pi t/6} \leq \|v(t)\|_{L^2} \leq \|v_0\|_{L^2} e^{\pi t/6}, \quad t > 0.$$

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Conclusion: The peaked periodic wave is *linearly unstable*.

Spectral instability of the peaked periodic wave: $p = 1$

Back to the spectral problem

$$\lambda v = Av := \partial_z \left[(c_* - U_*) v \right] + \partial_z^{-1} v,$$

with

$$\text{dom}(A) = \left\{ v \in \dot{L}^2_{\text{per}} : \quad \partial_z \left[(c_* - U_*) v \right] \in \dot{L}^2_{\text{per}} \right\}.$$

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Theorem (Geyer & P., 2019)

$$\sigma(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{6} \leq \text{Re}(\lambda) \leq \frac{\pi}{6} \right\}.$$

Truncated spectral problem

It is natural to consider the truncated spectral problem

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Lemma

Let $A : \mathrm{dom}(A) \subset X \to X$ *and* $A_0 : \mathrm{dom}(A_0) \subset X \to X$ *be linear operators on Hilbert space* X *with the same domain* $\mathrm{dom}(A_0) = \mathrm{dom}(A)$ *such that* $A - A_0 = K$ *is a compact operator in* X. *Assume that the intersections* $\sigma_{\rm p}(A) \cap \rho(A_0)$ *and* $\sigma_{\rm p}(A_0) \cap \rho(A)$ *are empty. Then,* $\sigma(A) = \sigma(A_0)$.

Spectrum of the truncated problem

We want to compute the spectrum of the truncated problem:

$$\lambda v = A_0 v := \frac{1}{6}\partial_z \left[(\pi^2 - z^2) v(z)\right].$$

Transformation in characteristic variables,

$$\frac{dz}{d\xi} = \tfrac{1}{6}(\pi^2 - z^2) \quad \Rightarrow \quad z = \pi \tanh\left(\frac{\pi\xi}{6}\right),$$

maps it to

$$\mu w = B_0 w := \partial_y w(y) - \tanh(y) w(y), \quad y \in \mathbb{R},$$

with $\mu = 6\lambda/\pi$ and $\mathrm{dom}(B_0) = H^1(\mathbb{R}) \cap \dot{L}^2(\mathbb{R})$,

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No point spectrum, whereas the essential spectrum is located at:

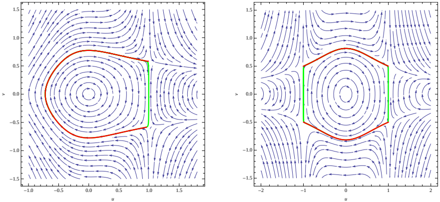
$$\sigma(B_0) = \{\mu \in \mathbb{C} : \quad -1 \leq \mathrm{Re}(\mu) \leq 1\}.$$

Summary

- Global solutions and wave breaking in the generalized reduced Ostrovsky equation

$$(u_t + u^p u_x)_x = u.$$

- Existence of smooth and peaked periodic waves



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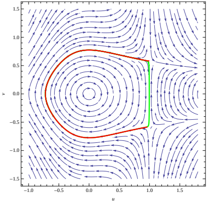

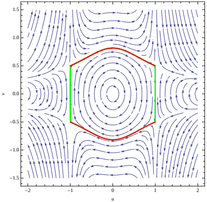

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Thank you! Questions???