Propagation Failure in the Discrete Nagumo Equation

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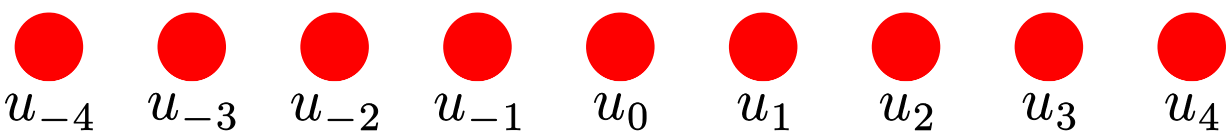
Lattice Differential Equations

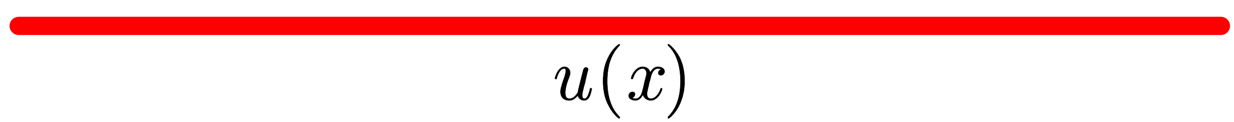
Lattice differential equations (LDEs) are ODEs indexed on a spatial lattice, e.g.

$$\frac{d}{dt}u_j(t) = \alpha\big(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\big) + f\big(u_j(t)\big), \qquad j \in \mathbb{Z}.$$



If $\alpha = h^{-2} \gg 1$, LDE can be seen as a discretization with step size h of PDE

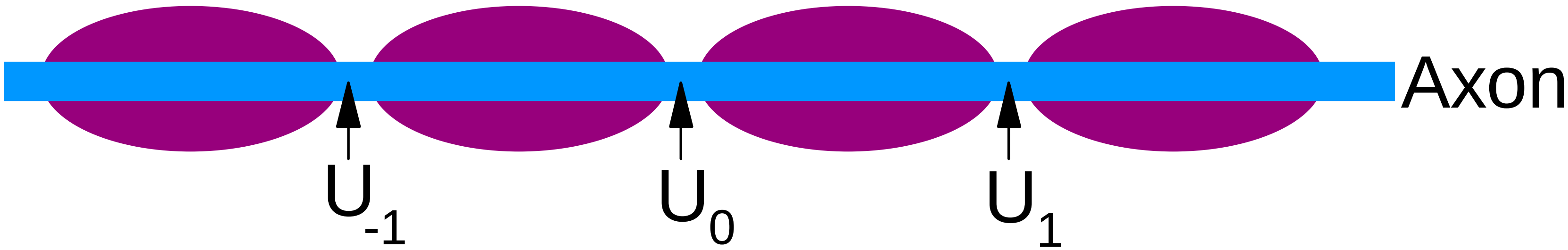
$$\partial_t u(t,x) = \partial_x^2 u(t,x) + f\big(u(t,x)\big), \qquad x \in \mathbb{R}.$$



- Many physical models have a discrete spatial structure $\to$ LDEs.
- Main theme: qualitative differences between PDEs and LDEs.

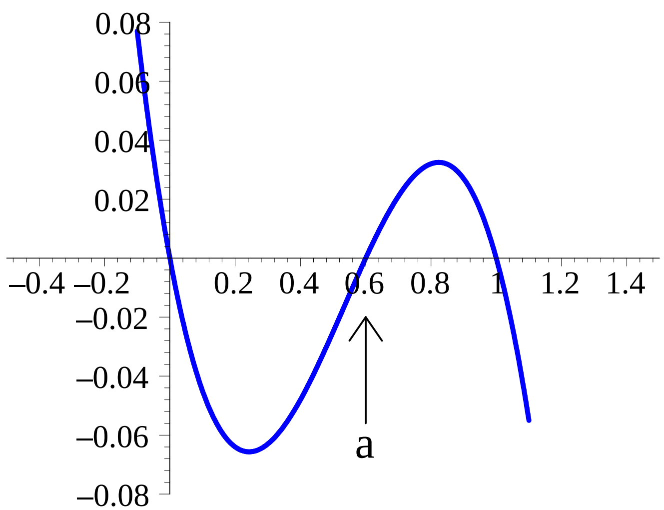
Signal Propagation through Nerves

One is interested in the potential U_j at the node sites.



Signals appear to "hop" from one node to the next [Lillie, 1925].
Ignoring recovery, one arrives at the LDE, called the discrete Nagumo equation [Keener and Sneyd, 1998]

$$\frac{d}{dt}U_j(t) \quad = \quad U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a), \qquad j \in \mathbb{Z}.$$



Bistable nonlinearity g given by

$$g(u; a) = u(a - u)(u - 1).$$

Traveling front solutions

In the continuum limit, the discrete Nagumo equation becomes the continuous Nagumo equation,

$$\partial_t u = \partial_x^2 u + u(a-u)(u-1).$$

Travelling wave $u(x,t) = \phi(x+ct)$ satisfies:

$$c\phi'(\xi) \quad = \quad \phi''(\xi) + \phi(\xi)\big(a-\phi(\xi)\big)\big(\phi(\xi)-1\big).$$

We are interested in the front solutions connecting stable equilibrium states 0 and 1 (heteroclinic orbits). These solutions satisfy the boundary conditions,

$$\lim_{\xi\to-\infty}\phi(\xi) = 0, \qquad \lim_{\xi\to+\infty}\phi(\xi) = 1.$$

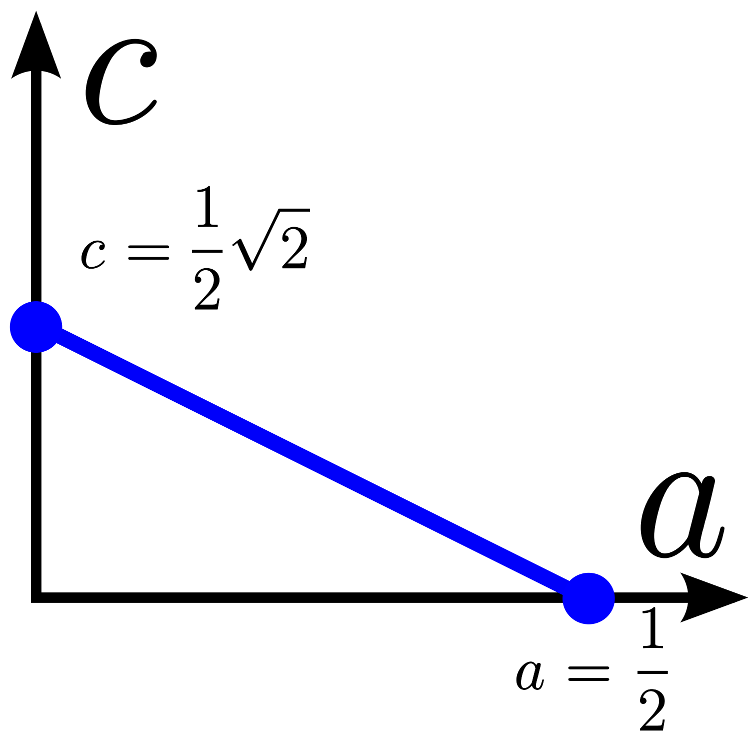
Exact solutions

Recall the travelling wave ODE

$$c\phi'(\xi) = \phi''(\xi) + \phi(\xi)\big(a - \phi(\xi)\big)\big(\phi(\xi) - 1\big),$$

subject to

$$\lim_{\xi \to -\infty} \phi(\xi) = 0, \quad \lim_{\xi \to +\infty} \phi(\xi) = 1.$$



Explicit solutions available:

$$\begin{aligned} \phi(\xi) &= \tfrac{1}{2} + \tfrac{1}{2}\tanh\left(\tfrac{1}{4}\sqrt{2}\,\xi\right), \\ c(a) &= \tfrac{1}{\sqrt{2}}(1 - 2a). \end{aligned}$$

Travelling fronts in LDE

Back to the Nagumo LDE

$$\frac{d}{dt}U_j(t) = \frac{1}{h^2}\left[U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)\right] + U_j(t)(a - U_j(t))(U_j(t) - 1),\ j \in \mathbb{Z}.$$

Travelling front solutions $U_j(t) = \phi(j + ct)$ must satisfy:

$$c\phi'(\xi) = \frac{1}{h^2}\left[\phi(\xi + h) + \phi(\xi - h) - 2\phi(\xi)\right] + \phi(\xi)(a - \phi(\xi))(\phi(\xi) - 1),$$

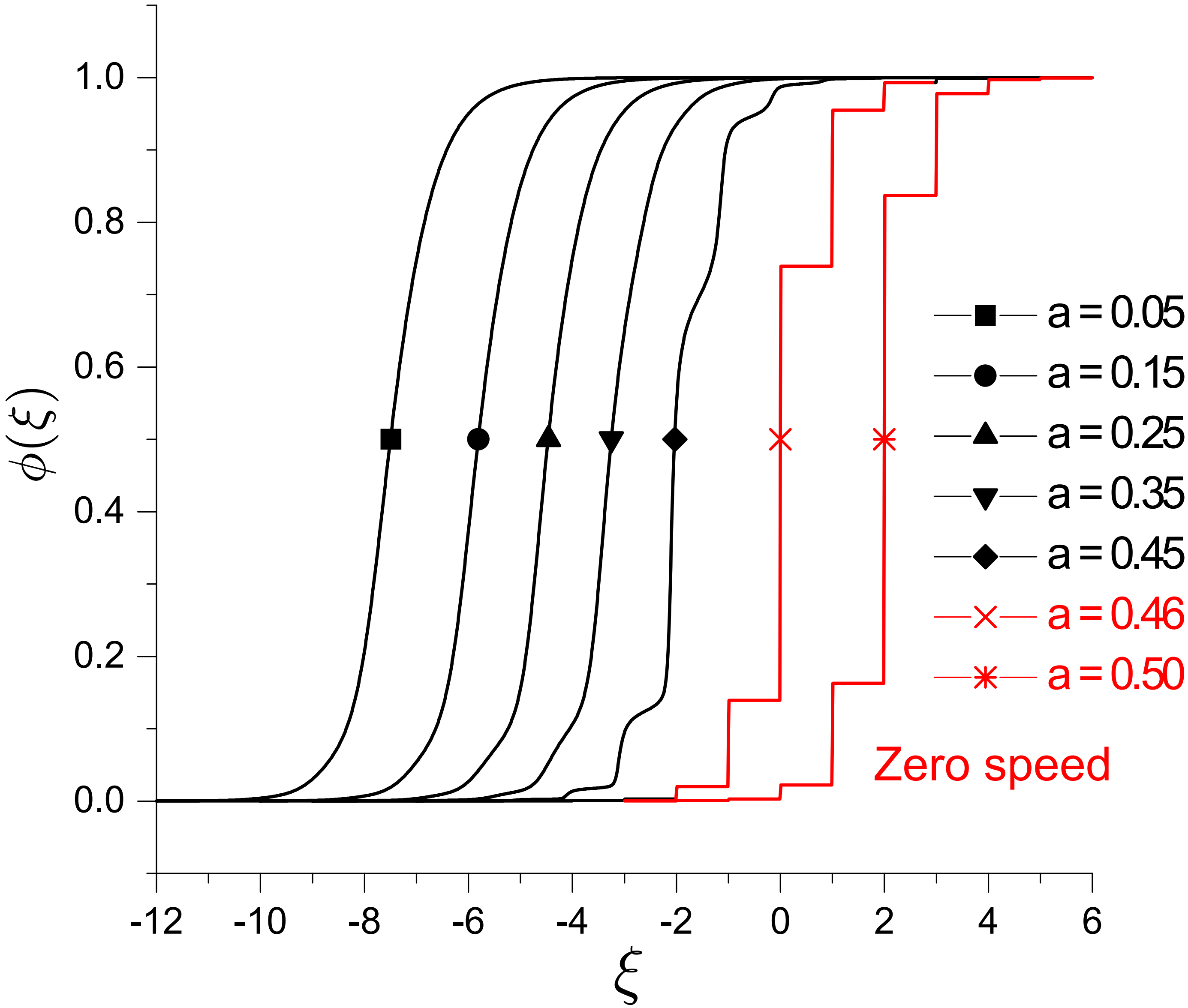
subject to

$$\lim_{\xi \to -\infty} \phi(\xi) = 0, \quad \lim_{\xi \to +\infty} \phi(\xi) = 1.$$

- When $c \neq 0$, this is a differential advance-delay equation.
- When $c = 0$, this is an advance-delay equation.
- The limit $c \to 0$ is a singular perturbation theory.

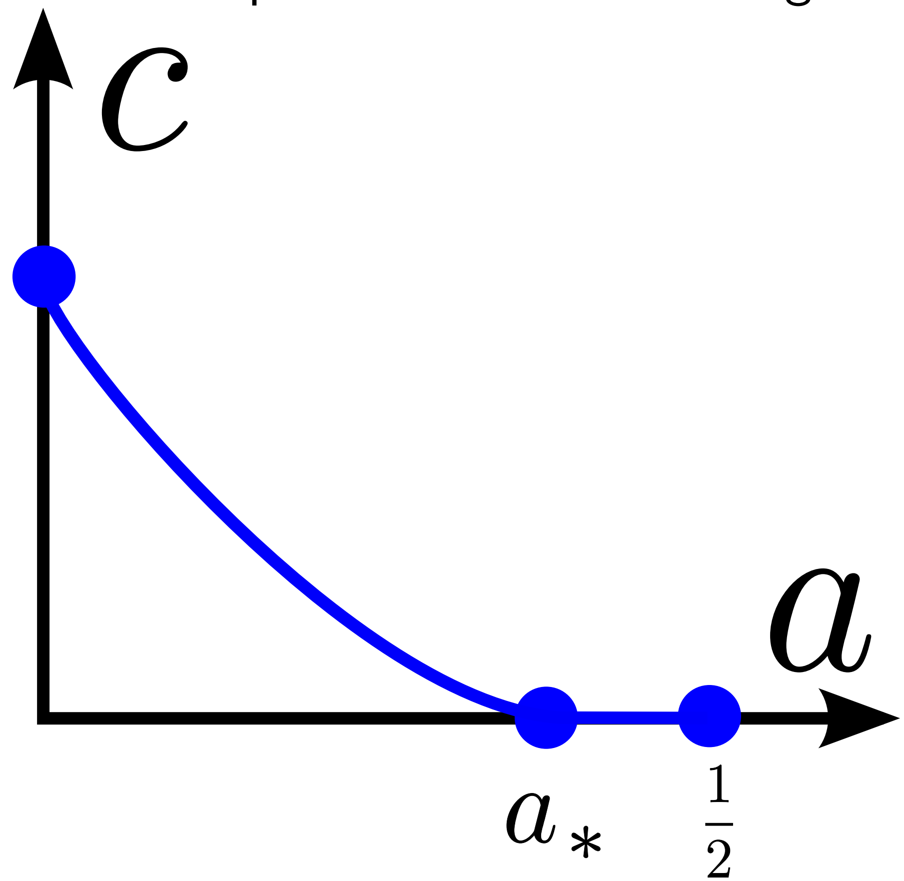
Discrete Nagumo LDE - Propagation failure

Travelling waves for the discrete Nagumo LDE connecting $0 \to 1$.



Propagation failure

Typical wave speed c versus a plot for the discrete Nagumo LDE:



We can have either $a_* = \frac{1}{2}$ or $a_* < \frac{1}{2}$.

If $a_* < \frac{1}{2}$, we say that LDE suffers from propagation failure.

Propagation failure widely studied; pioneed by [Keener].

Propagation failure

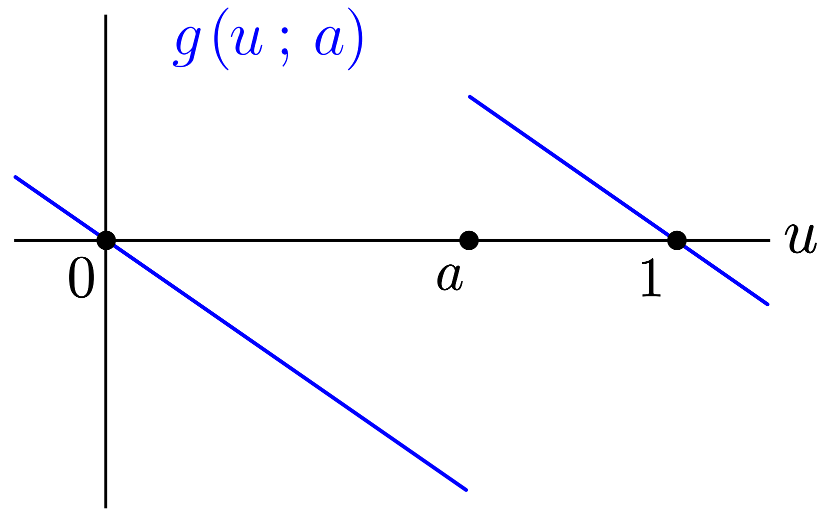
Consider travelling wave MFDE with saw-tooth nonlinearity

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi+h)+\phi(\xi-h)-2\phi(\xi)] + g\big(\phi(\xi);a\big),$$

subject to

$$\lim_{\xi\to-\infty}\phi(\xi)=0, \quad \lim_{\xi\to+\infty}\phi(\xi)=1.$$

Propagation failure for all $h>0$
[Cahn, Mallet-Paret, Van Vleck] (1999)



Linear analysis with Fourier series.

Propagation failure

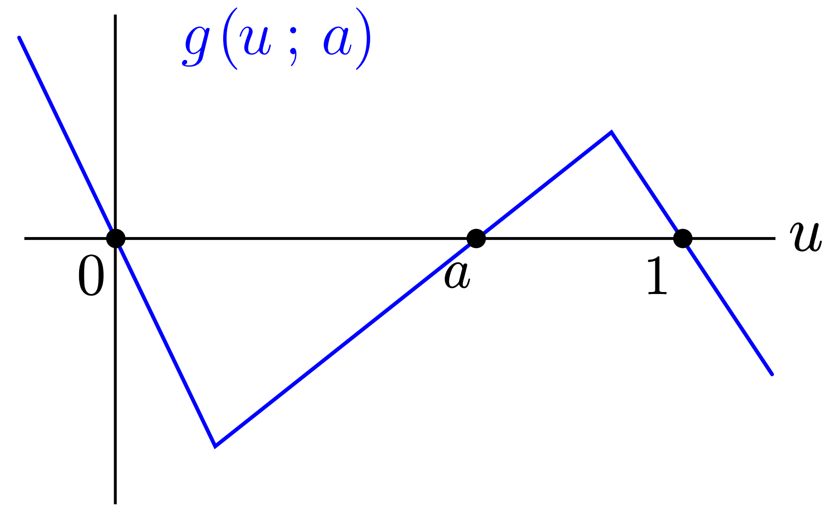
Consider travelling wave MFDE with zig-zag bistable nonlinearity

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi+h) + \phi(\xi-h) - 2\phi(\xi)] + g\big(\phi(\xi); a\big),$$

subject to

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \quad \lim_{\xi\to+\infty} \phi(\xi) = 1.$$

There exist countably many h for which there is no propagation failure.
[Elmer] (2006)



Propagation failure for the Klein–Gordon equation

In a similar context of the Klein-Gordon equation,

$$u_{tt} = u_{xx} + g(u\,;\,a),$$

many researchers were looking for other discretizations of g that admit a continuous ("translationally invariant") branch of stationary solutions [Speight](1999); [Kevrekidis](2003); [Barashenkov, Oxtoby, Pelinovsky](2005); [Dmitriev, Kevrekidis, Yoshikawa](2005).

Main Question: Does the existence of continuous ("translationally invariant") stationary solutions imply the existence of continuously differentiable traveling solutions?

The Answer is NO for the discrete Klein–Gordon equation.

Example

The discrete Nagumo equation

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi+h) + \phi(\xi-h) - 2\phi(\xi)] + g\big(\phi(\xi); a\big)$$

$$g(u;a) = \frac{2(1-u)u(u-a)(1+h^2(1+au))}{(1+h^2(1-u)u)(1+h^2(1-a)a)},$$

admits an exact traveling front solution,

$$\phi(z) = \frac{1}{2}(1+\tanh(bz-s)), \quad b = \frac{arcsinh(h)}{h}, \quad c = \frac{2a-1}{b(1+h^2(1-a)a)}, \quad s \in \mathbb{R}.$$

If $a = \frac{1}{2}$, then $c = 0$, and the stationary front is "translationally invariant" $\phi(z) = \tanh(bz - s)$ with arbitrary parameter $s \in \mathbb{R}$ (the same for KG equation).

We can see that stationary front becomes a traveling front without a propagation failure.

Question: Is this a coincidence?

Formulation of the problem

Recall the differential advance-delay equation for travelling waves:

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi+h) + \phi(\xi-h) - 2\phi(\xi)] + \phi(\xi)\big(a - \phi(\xi)\big)\big(\phi(\xi) - 1\big).$$

When $c = 0$, we can restrict to $\xi \in \mathbb{Z}$ and obtain a difference equation.

With $p_j = \phi(j)$ and $r_j = \phi(j+1)$, we find

$$\begin{aligned} p_{j+1} &= r_j \\ r_{j+1} &= -p_j + 2r_j - h^2 r_j (r_j - a)(1 - r_j). \end{aligned}$$

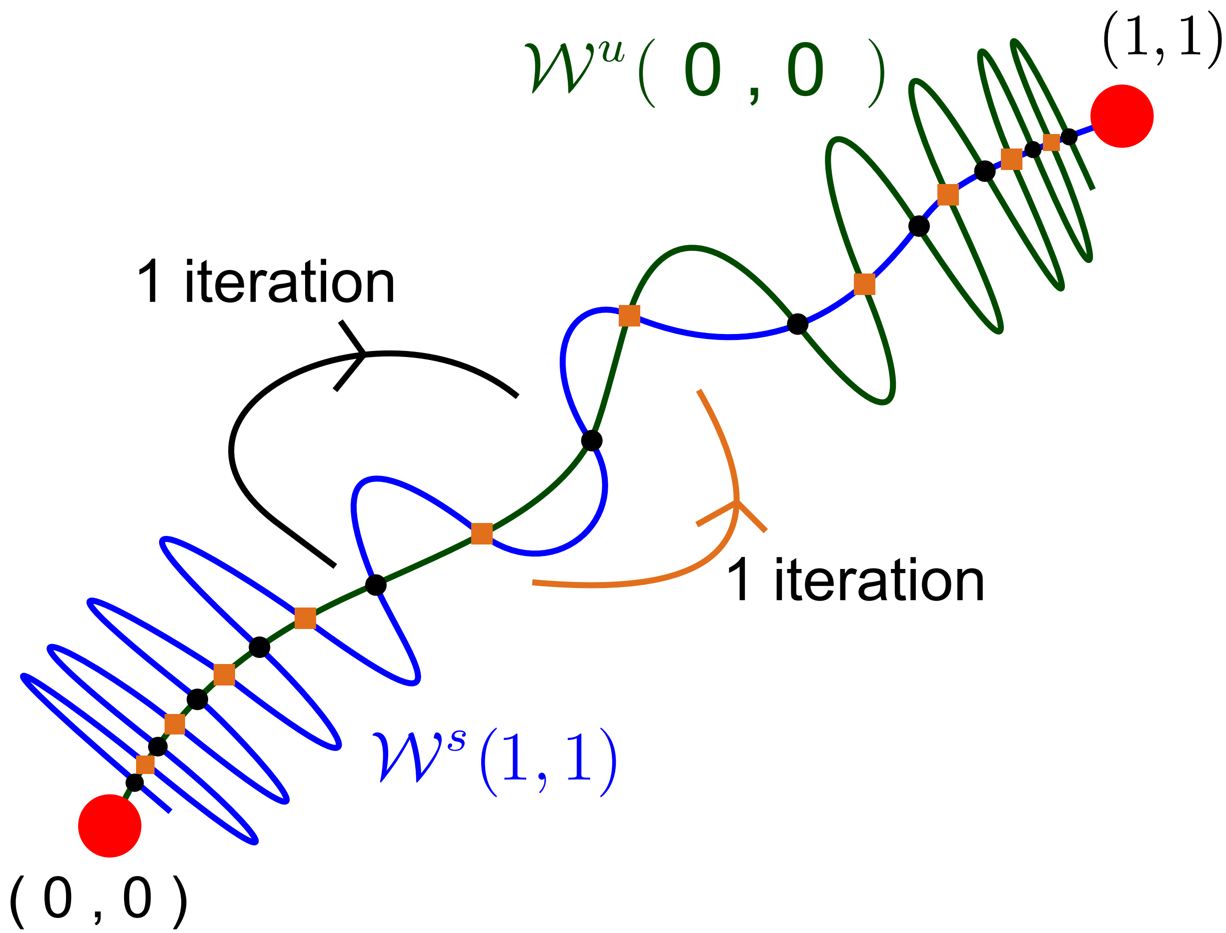
Two fixed point $(0,0)$ and $(1,1)$ are saddles. Generally, two heteroclinic orbits exist for $a = \frac{1}{2}$ (symmetric case):

$$p^{(s)}_{-j} = -p^{(s)}_j, \quad p^{(b)}_{-j+1} = -p^{(b)}_j,$$

called site-symmetric and bond-symmetric fronts.

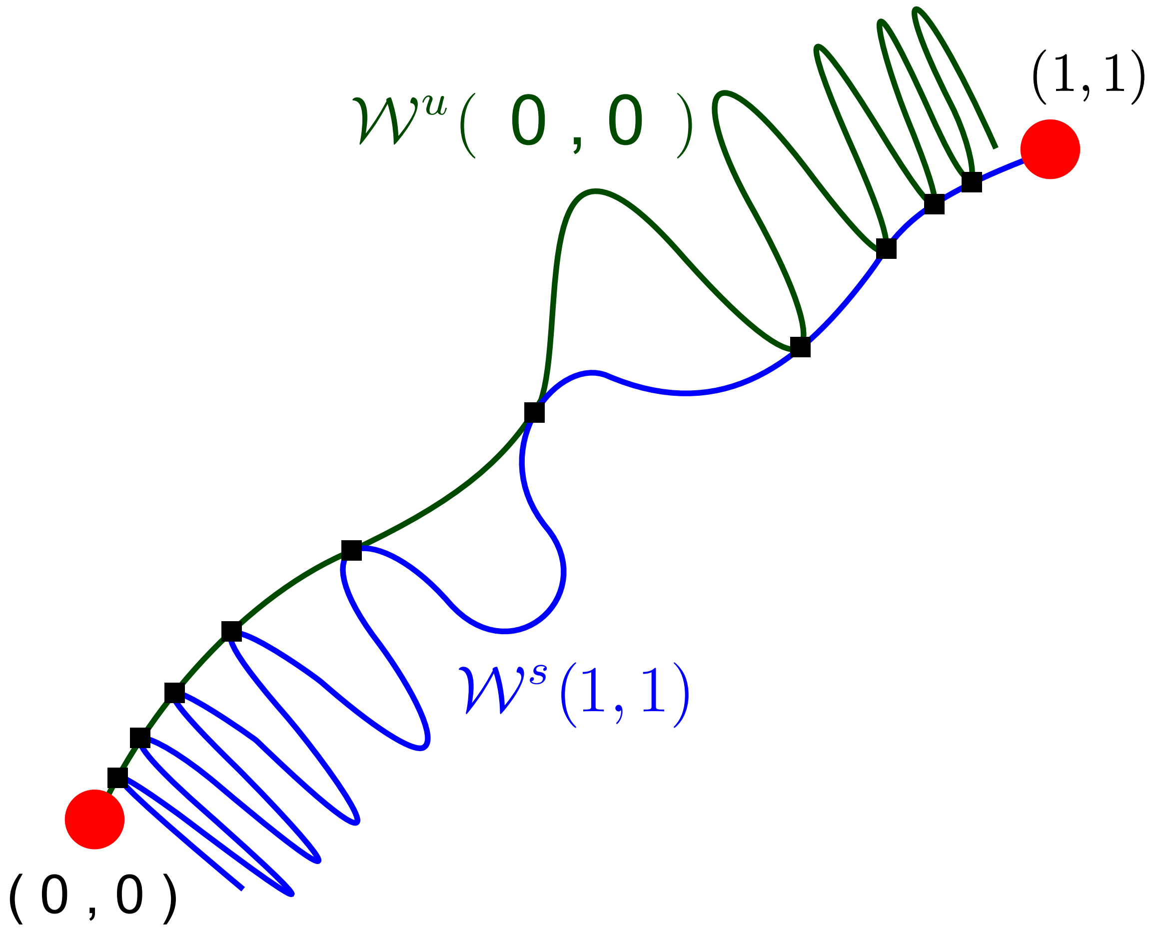
Formulation of the problem

For $a = \frac{1}{2}$, site-symmetric (orange) and bond-symmetric (black) solutions:



Formulation of the problem

For $a < \frac{1}{2}$, the distance between nodes decreases. At $a = a_* < \frac{1}{2}$, two branches of stationary front solutions coincide and annihilate via a saddle-node bifurcation.



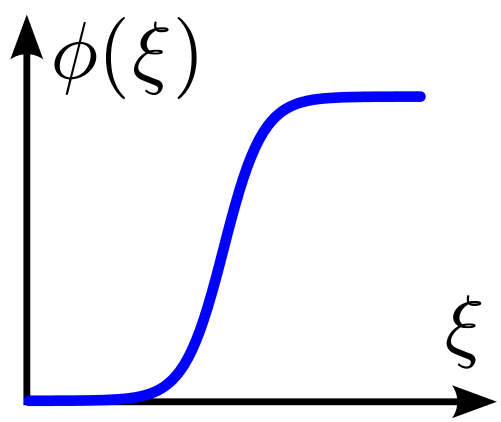
Formulation of the problem

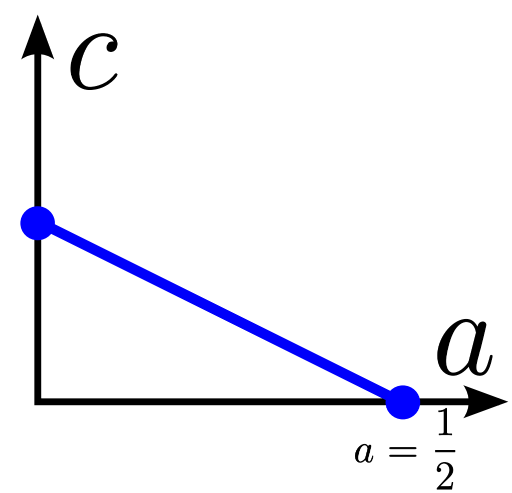
Special discretizations of g may also involve multiple lattice sites:

$$\frac{d}{dt}U_j = \frac{1}{h^2}[U_{j-1} + U_{j+1} - 2U_j] + \frac{1}{2}U_j(U_{j+1} + U_{j-1} - 2a)(1 - U_j).$$

Explicit solutions available:

$$U_j(t) = \frac{1}{2} + \frac{1}{2}\tanh\Big(\operatorname{arcsinh}(\frac{1}{4}\sqrt{2}\,h)(j + ct)\Big), \qquad c(a) = \frac{(1-2a)}{4\operatorname{arcsinh}(\frac{1}{4}\sqrt{2}\,h)}.$$

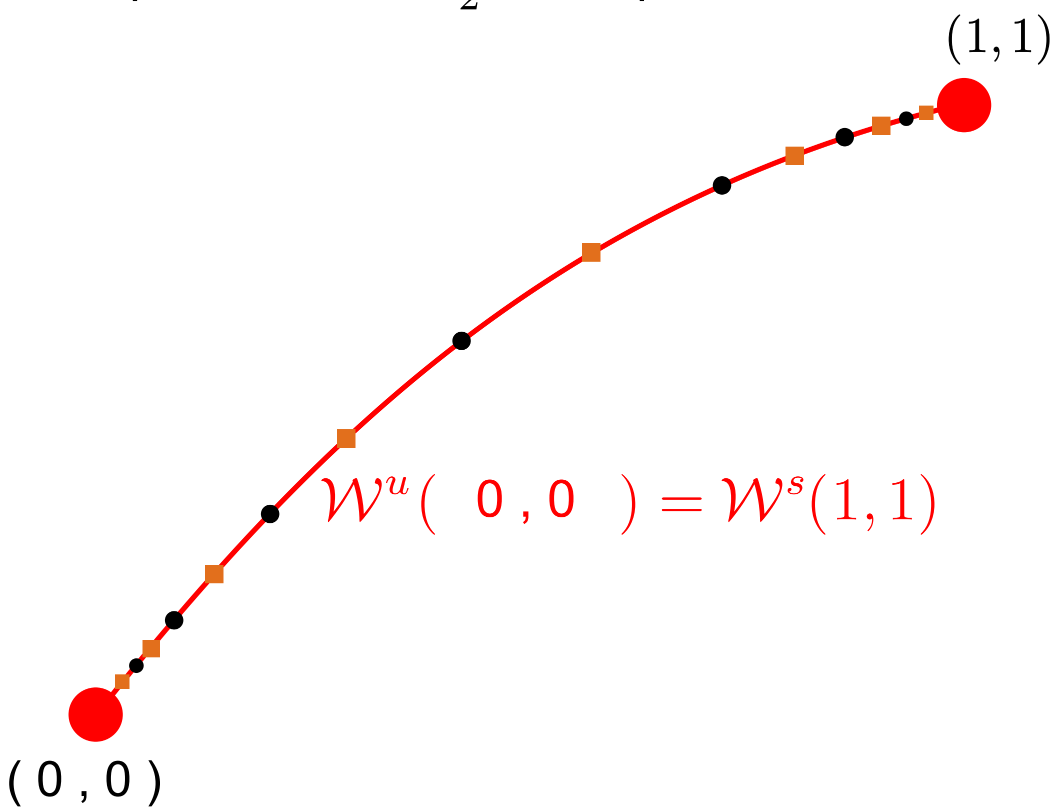




No propagation failure; smooth wave profile.

Formulation of the problem

Smooth standing wave profile at $a = \frac{1}{2}$ correspond to:



Site-symmetric and bond-symmetric solutions are connected by a continuous branch of “translationally invariant” standing waves.

Main Question: What happens to manifolds when $a \neq \frac{1}{2}$?

Do intersections disappear (no prop failure) or survive (prop failure)?

Lattice point of view

Let us write LDE as:

$$\frac{d}{dt}U(t) = \mathcal{F}(U(t)\,;\,a),$$

with $U(t) \in \ell^\infty$ and $\mathcal{F} : \ell^\infty \times [0,1] \to \ell^\infty$.

Travelling waves $U_j(t) = \phi(j + ct)$ satisfy the differential advance-delay equation,

$$c\phi'(\xi) = \mathcal{G}\Big(\phi(\xi-1), \phi(\xi), \phi(\xi+1)\,;\,a\Big)$$

Suppose at $a = \frac{1}{2}$ we have a smooth solution $p(\xi)$ to

$$0 = \mathcal{G}\Big(p(\xi-1), p(\xi), p(\xi+1)\,;\,a = \frac{1}{2}\Big), \qquad \xi \in \mathbb{R}.$$

Then for every $\vartheta \in \mathbb{R}$, we have equilibrium solution $p^{(\vartheta)} \in \ell^\infty$ to our LDE:

$$\mathcal{F}\left(p^{(\vartheta)}\,;\,\frac{1}{2}\right) = 0, \qquad p_j^{(\vartheta)} = p(\vartheta + j).$$

Invariant Manifold

Recall $p^{(\vartheta)} \in \ell^\infty$ with $p_j^{(\vartheta)} = p(\vartheta + j)$.

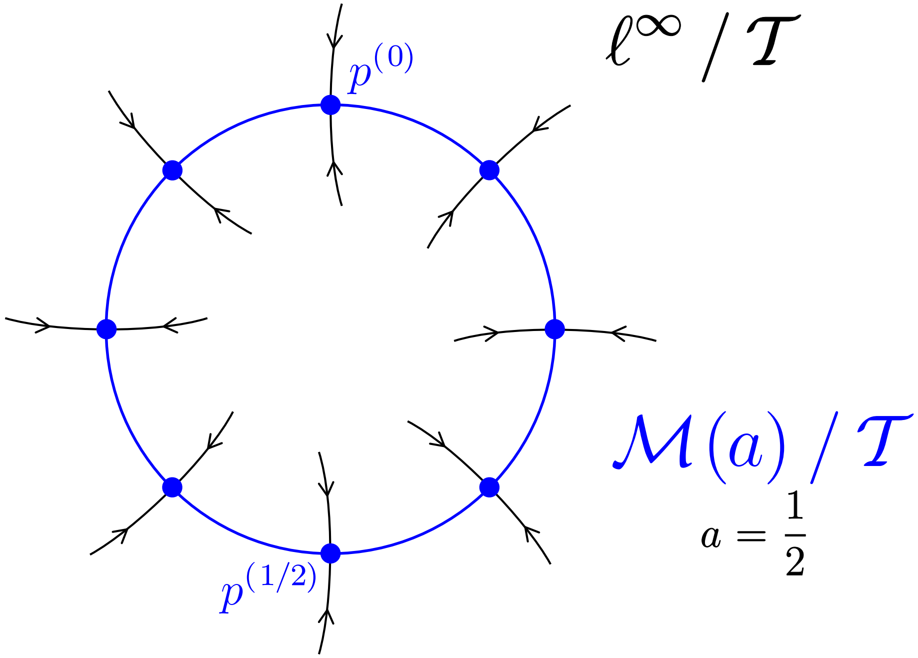
Notice that

$$p^{(\vartheta)} = \mathcal{T} p^{(\vartheta+1)},$$

where $\mathcal{T} : \ell^\infty \to \ell^\infty$ is right-shift operator $(\mathcal{T}u)_j = u_{j-1}$.

Combining these equilibria gives a smooth manifold

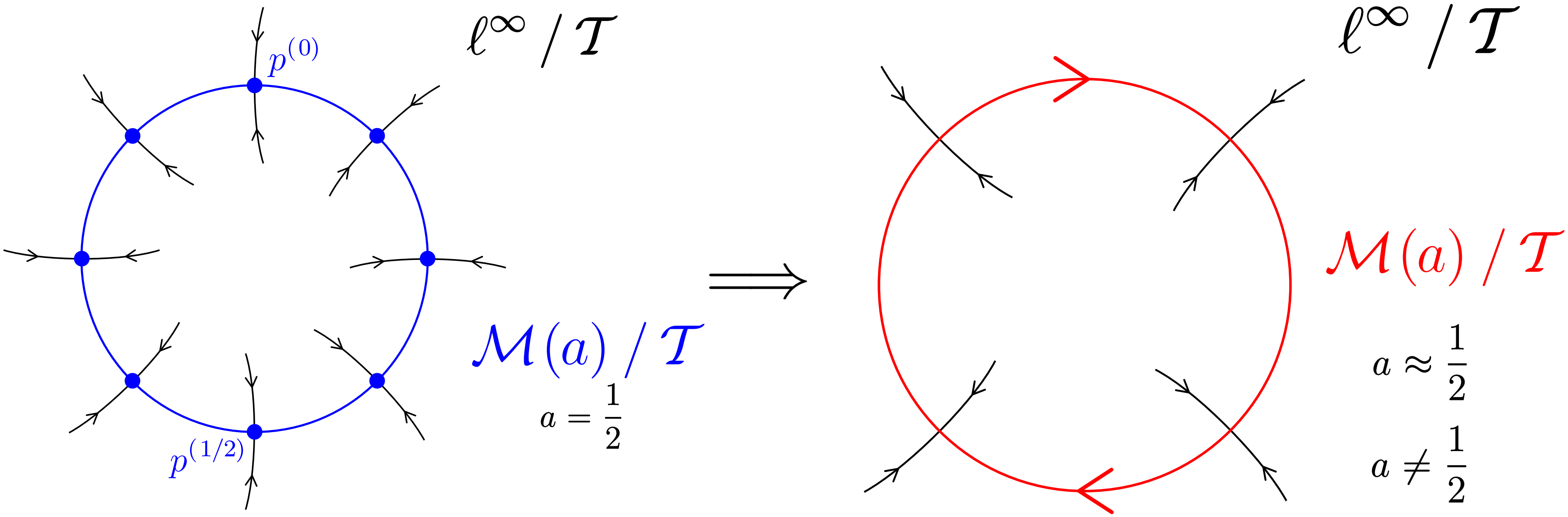
$$\mathcal{M}\left(a = \frac{1}{2}\right) = \{p^{(\vartheta)}\}_{\vartheta \in \mathbb{R}}.$$



Based on spectral stability of equilibria $p^{(\vartheta)}$ [Chow, Mallet-Paret, Shen, 1998] and comparison principles, we can prove that the manifold $\mathcal{M}(a = \frac{1}{2})$ is normally hyperbolic.

Invariant Manifold - Scenario #1

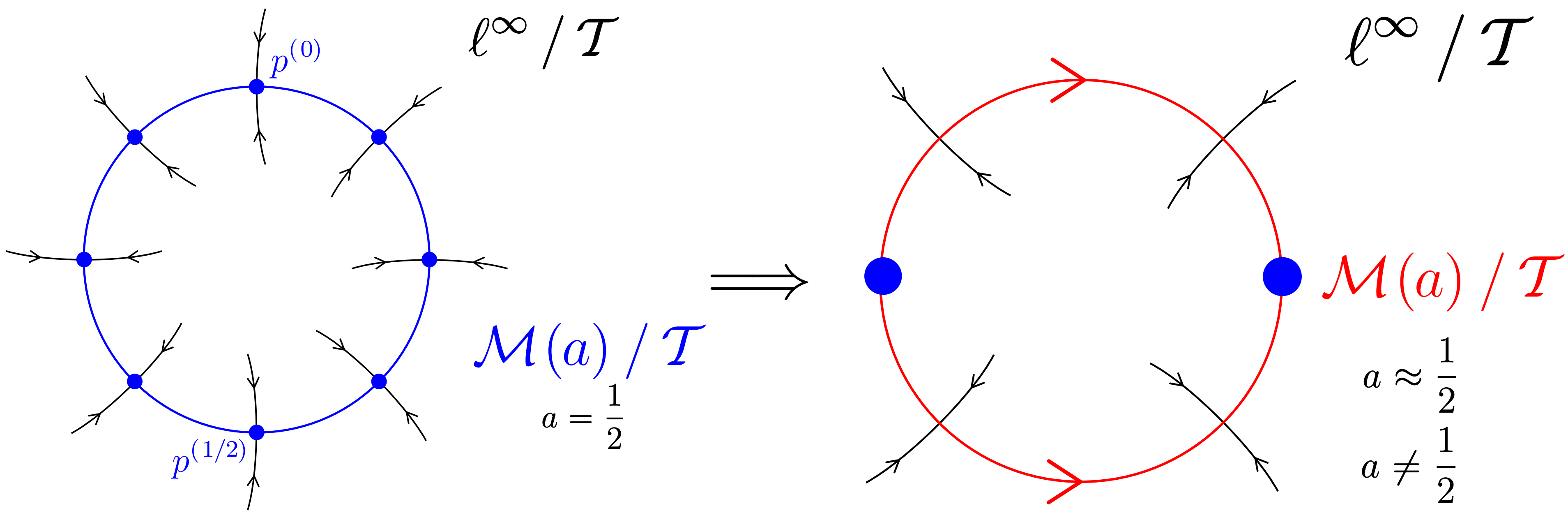
Possible scenario #1 for persistence of $\mathcal{M}(a)$ with $a \neq \frac{1}{2}$:



No equilibria survive; $\mathcal{M}(a)$ is orbit of travelling wave. No Propagation Failure.

Invariant Manifold - Scenario #2

Possible scenario #2 for persistence of $\mathcal{M}(a)$ with $a \neq \frac{1}{2}$:



One or more equilibria survive. Propagation Failure.

Dynamics at $\mathcal{M}(a)$

Angular coordinate θ measures position along $\mathcal{M}(a)$. Dynamics at $\mathcal{M}(a)$ for $a \approx \frac{1}{2}$ is given by

$$\frac{d}{dt}\theta = (a - \frac{1}{2})\Psi(\theta) + O\big(\left| a - \frac{1}{2} \right|^2 \big),$$

in which $\Psi(\theta)$ given by

$$\Psi(\vartheta) = \sum_{j \in \mathbb{Z}} q_j^{(\vartheta)} \partial_a \mathcal{G}\Big(p_{j-1}^{(\vartheta)}, p_j^{(\vartheta)}, p_{j+1}^{(\vartheta)} \,;\, a = \frac{1}{2} \Big).$$

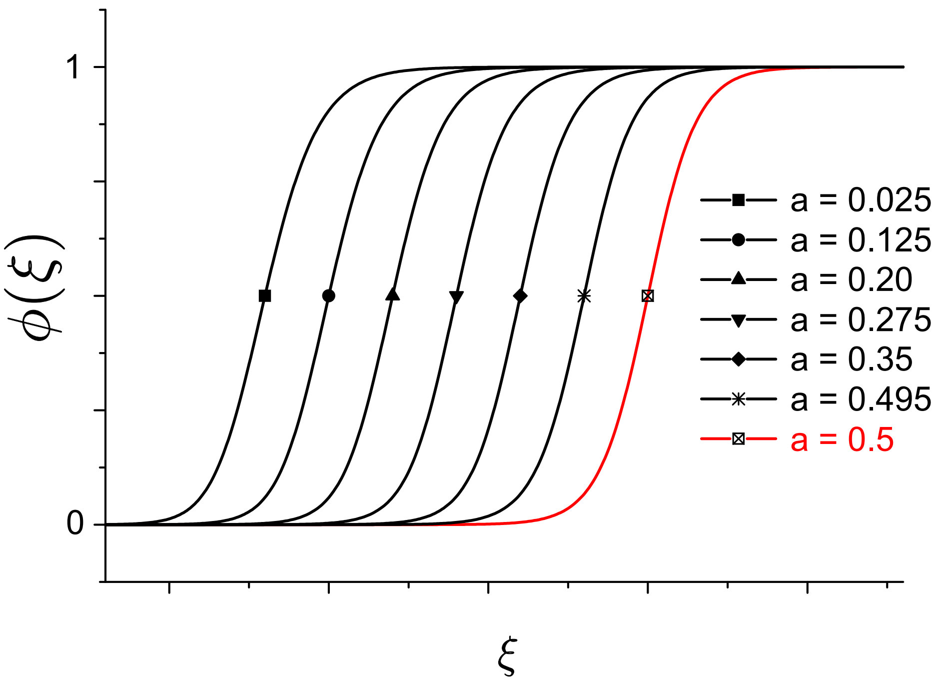
Here $q^{(\vartheta)}$ is adjoint eigenvector; i.e. solves $L^{(\vartheta)*} q^{(\vartheta)} = 0$.

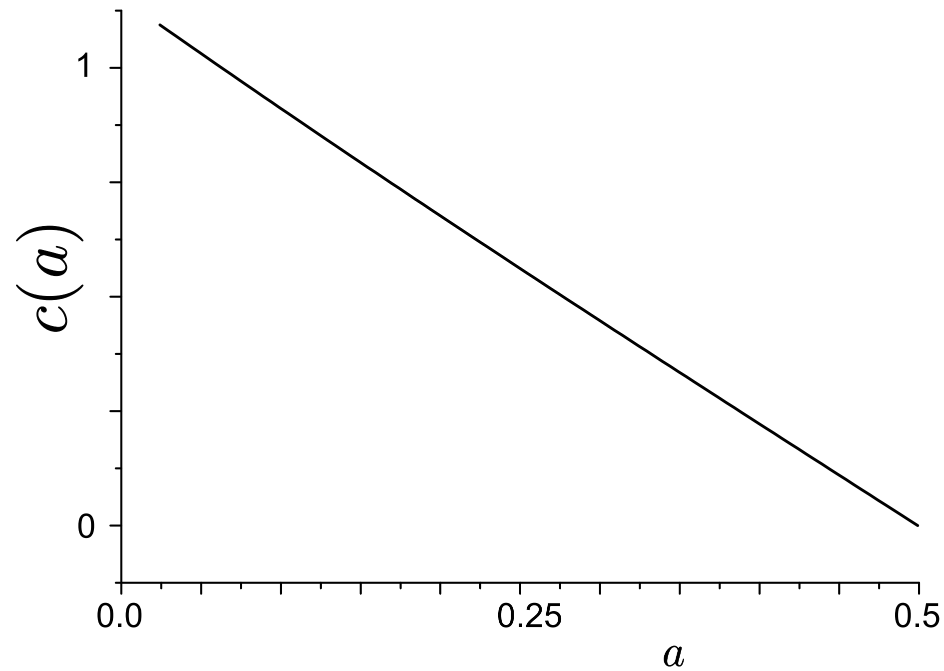
Known: $q_j^{(\vartheta)} > 0$ for all $j \in \mathbb{Z}$ and $\vartheta \in \mathbb{R}$. So $\partial_a \mathcal{G} < 0$ guarantees no prop failure.

Example 1

No prop failure for LDE

$$\frac{d}{dt}u_j \quad = \quad u_{j-1} + u_{j+1} - 2u_j + (u_j - a)\Big(u_{j-1}(1 - u_{j+1}) + u_{j+1}(1 - u_{j-1})\Big)$$

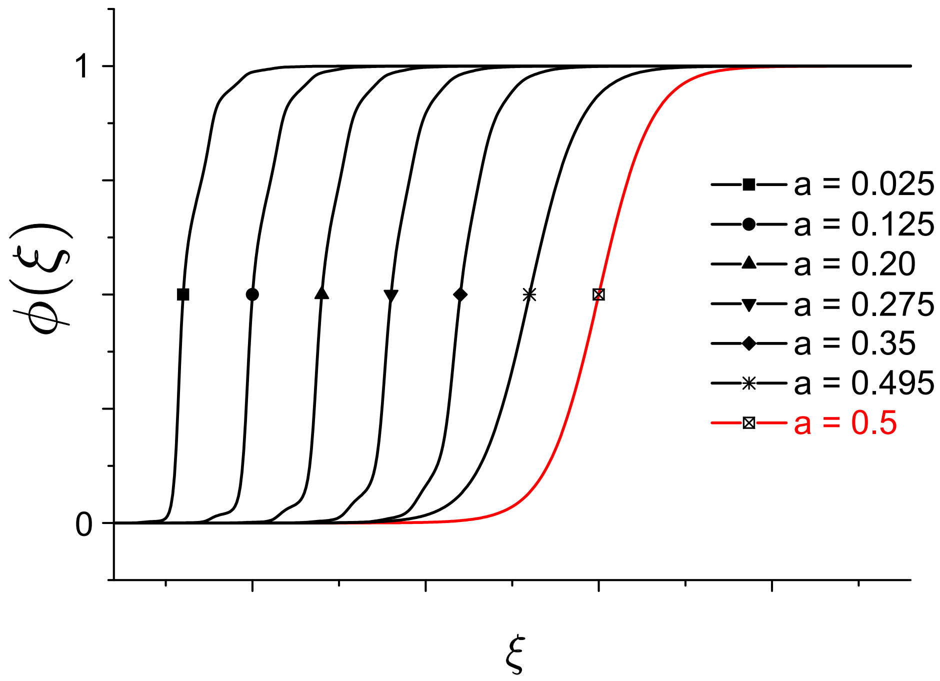


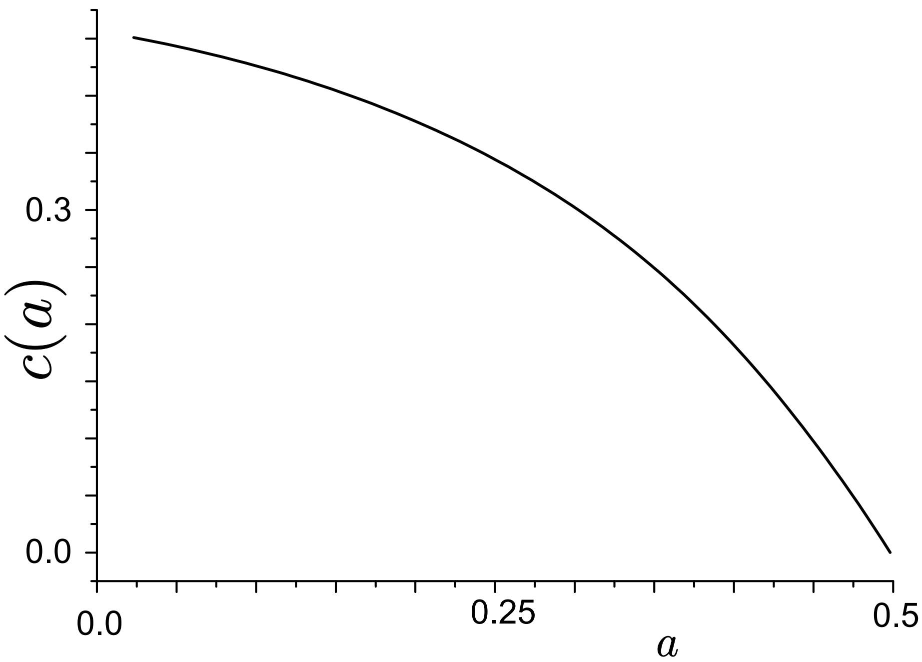


Example 2

No prop failure for LDE

$$\frac{d}{dt}u_j = u_{j-1} + u_{j+1} - 2u_j + (u_j - a)\Big(u_{j-1}(1 - u_{j+1}) + u_{j+1}(1 - u_{j-1})\Big)$$
$$-\tfrac{5}{4}(a - \tfrac{1}{2})\sin(2\pi u_j).$$



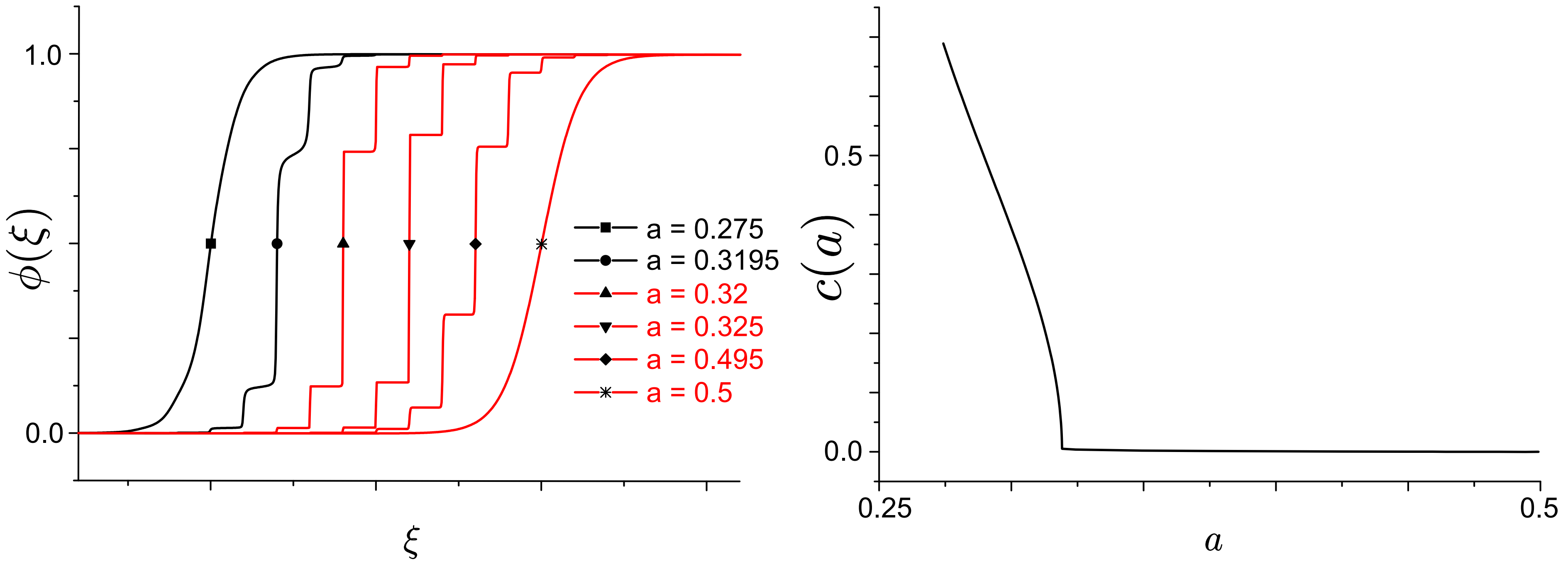


Here $\partial_a \mathcal{G}$ may have both signs, but (numerically) $\Psi(\theta) < 0$ for all θ.

Example 3

Do have prop failure for LDE

$$\begin{aligned}\frac{d}{dt}u_j &= u_{j-1} + u_{j+1} - 2u_j + 4u_j(1-u_j)(u_{j-1}+u_{j+1}-2a)\\ &\quad -5(a-\tfrac{1}{2})\sin(2\pi u_j)(\tfrac{6}{5}+\tfrac{8}{5}u_j).\end{aligned}$$



Numerically computed: $\Psi(\theta = 0) < 0 < \Psi(\theta = \frac{1}{2})$.

Differential advance-delay equation point of view

Let us write the traveling wave problem as

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi+h) + \phi(\xi-h) - 2\phi(\xi)] + g\big(\phi(\xi);a\big),$$

with $\phi(\xi) \in H^1(\mathbb{R})$ and $g : H^1(\mathbb{R}) \times [0,1] \to H^1(\mathbb{R})$.

Differential advance-delay operator $L_c : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is

$$(L_c\psi)(z) := -c\psi'(\xi) + \frac{1}{h^2}[\psi(\xi+h) + \psi(\xi-h) - 2\psi(\xi)] + g'\big(\phi(\xi);a\big)\psi(\xi),$$

Under the same assumptions, we have at $a = \frac{1}{2}$,

$$Ker(L_0) = span\left\{\varphi'(\xi)e^{i\kappa m\xi}\right\}_{m\in\mathbb{Z}}, \quad \kappa = \frac{2\pi}{h},$$

where $\varphi(\xi)$ is the stationary front solution for $c = 0$ and $a = \frac{1}{2}$.

D.P., Journal of Dynamics and Differential Equations **23**, 167–183 (2011)

Differential advance-delay equation point of view

Perturbation theory for small $c \neq 0$ and $a = \frac{1}{2}$ gives:

a unique real eigenvalue λ_c such that

$$\lambda_c = \mathcal{O}(c^2) \quad \text{as} \quad c \to 0.$$

the corresponding eigenfunction $\chi_c \in H^1(\mathbb{R})$ such that

$$\|\chi_c - \varphi'\|_{L^2} \geq C > 0 \quad \text{as} \quad c \to 0.$$

a countable set of simple eigenvalues

$$\lambda_c^{(m)} = \lambda_c - i\kappa m c, \quad \chi_c^{(m)}(\xi) = \chi_c(\xi) e^{i\kappa m \xi}, \quad m \in \mathbb{Z}.$$

Numerical approximations for small c

$$(L_0\psi)(z) := \frac{1}{h^2}[\psi(\xi+h)+\psi(\xi-h)-2\psi(\xi)]+\frac{2(2-3sech^2(b\xi)-h^2sech^4(b\xi))}{(1+h^2sech^2(b\xi))^2}\psi(\xi),$$

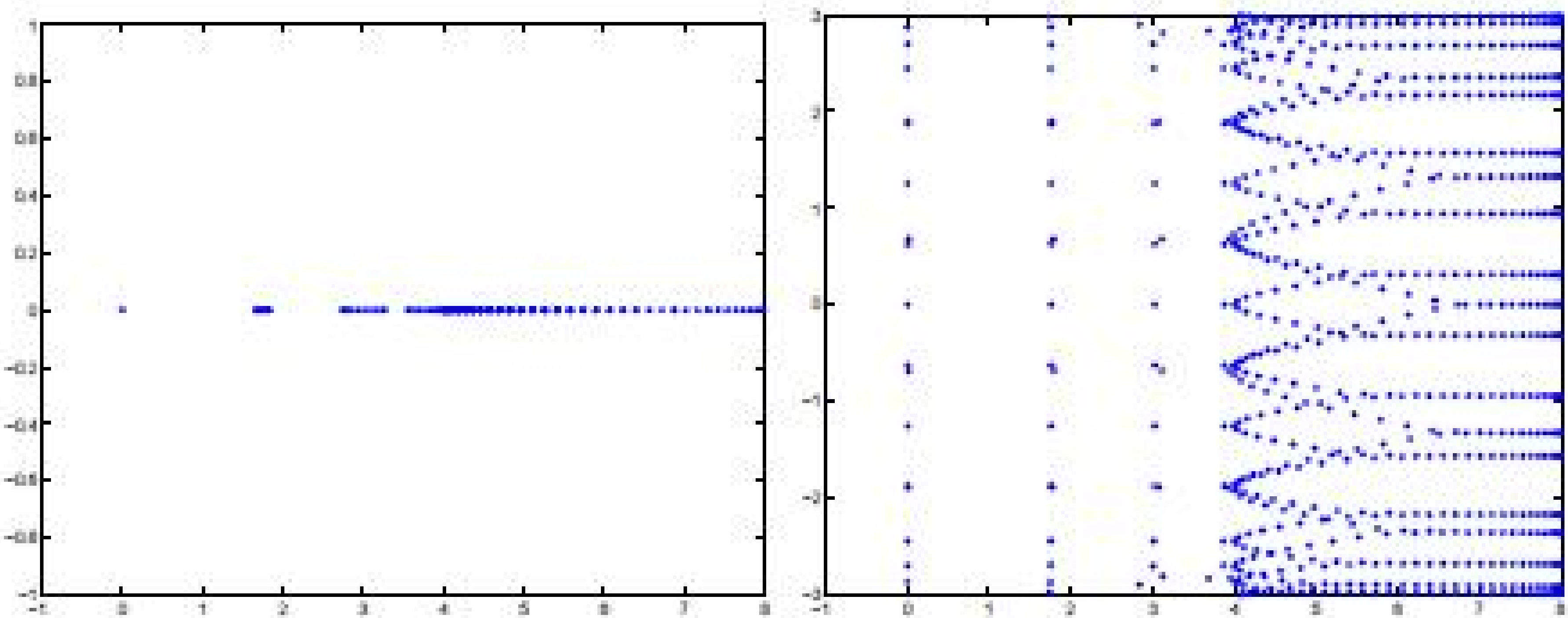


Figure 1: Numerical approximation of spectrum of L_c for $c = 0$ (left) and $c = 0.1$ (right).

Numerical approximations for small c

$$\lambda_c = \mathcal{O}(c^2), \quad \|\chi_c - \varphi'\|_{L^2} = \mathcal{O}(1), \quad \text{as} \quad c \to 0.$$

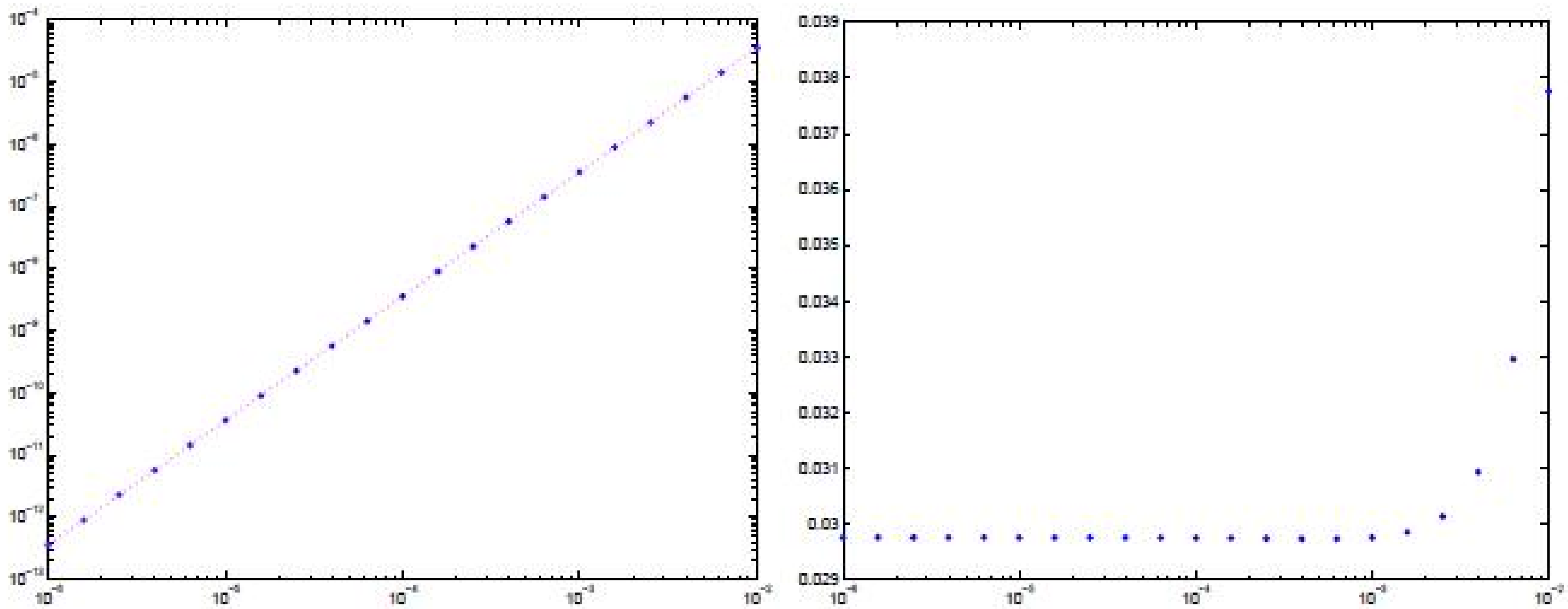


Figure 2: Left: convergence of the smallest eigenvalue of L_c as $c \to 0$. The dotted curve shows the power fit with $c^{1.9997}$. Right: the norm $\|\chi_c - \chi\|_{L^2}$ versus c for the corresponding eigenvector.

Numerical approximations for small c

$$(L_c - \lambda_c I)\psi = f_c : \quad \langle \theta_c, f_c \rangle_{L^2} = 0,$$

where θ_c is the eigenvector of the adjoint operator L_c^*.

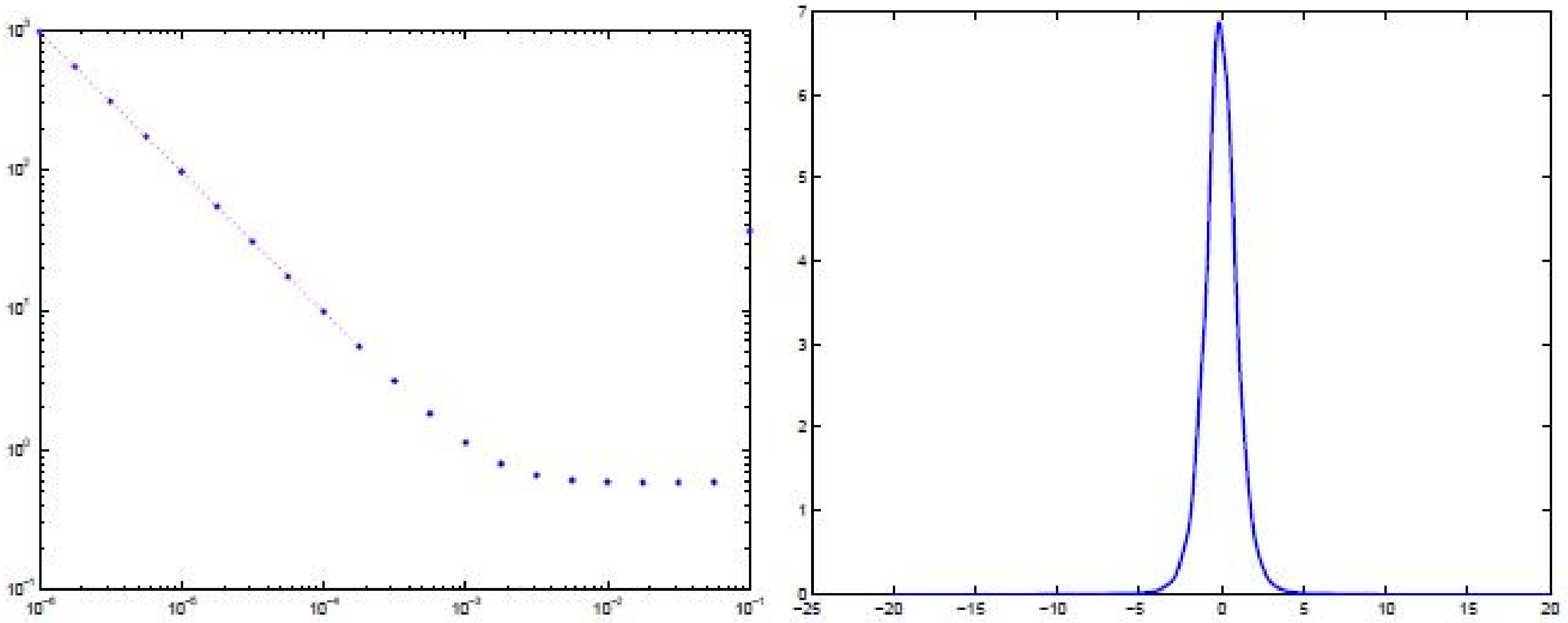


Figure 3: Left: the norm $\|\psi\|_{L^2}$ versus c. The dotted curve shows the power fit with $c^{-0.9993}$. Right: the solution $\psi(z)$ for $c = 0.1$.

Projection method

Differential advance-delay equation,

$$c\phi'(\xi) = \frac{1}{h^2}[\phi(\xi+h) + \phi(\xi-h) - 2\phi(\xi)] + g\big(\phi(\xi); a\big),$$

The decomposition

$$\phi(z) = \varphi(z) + \psi(z), \quad \langle \varphi', \psi \rangle_{L^2} = 0$$

is not sufficient because of the singular behavior $\|\psi\|_{H^1} = \mathcal{O}(c^{-1})$ as $c \to 0$.

If $\varphi(z)$ is a solution and $g(z+h) = g(z)$ is any C^1 function such that $\|g'\|_{L^\infty} < 1$, then $\tilde{\varphi}(\tilde{z})$ is also a solution of the advanced-delay equation with $c = 0$, where

$$z = \tilde{z} - g(\tilde{z}) \quad \Rightarrow \quad \frac{d\tilde{z}}{dz} = 1 + \sum_{m \in \mathbb{Z}} b_m e^{i\kappa m z}.$$

Coefficients $\{b_m\}_{m\in\mathbb{Z}}$ can be chosen to remove singular projections and to prove $c(a) = c_1(a - \frac{1}{2}) + \mathcal{O}\left(a - \frac{1}{2}\right)$.