Rogue waves, algebraic solitons, and breathers on the varying wave background in integrable models

Dmitry E. Pelinovsky

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada

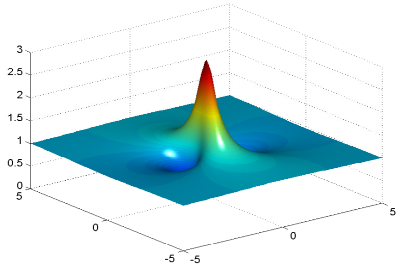
An important starting point: April 2016 Linz Austria

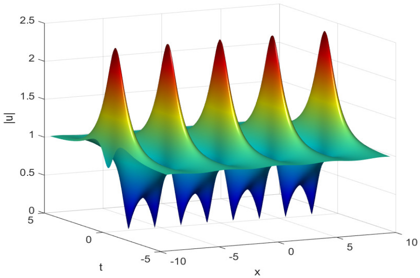
Question that has been asked in 2016

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0$$

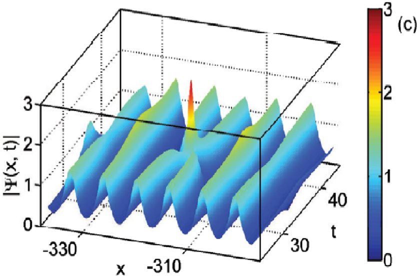
admits exact solutions for the rogue waves (Peregrine soliton, Akhmediev breather).

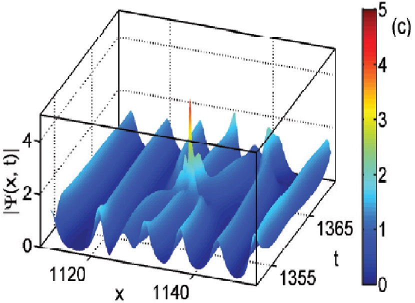




Is there an analogue of rogue waves on the varying backrgound?

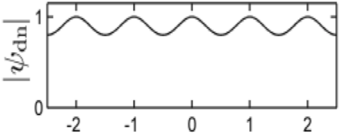
Numerical simulations in **D. Agafontsev–V.E. Zakharov (2016)** suggested that rogue waves play the major role in dynamics of periodic waves.

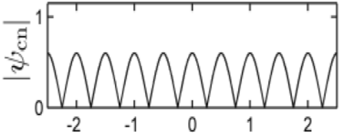




Simplest periodic wave backgrounds:

$$\psi_{\rm dn}(x,t) = {\rm dn}(x;k) e^{i(1-k^2/2)t}, \quad \psi_{\rm cn}(x,t) = k{\rm cn}(x;k) e^{i(k^2-1/2)t}.$$





Rogue waves on the periodic wave background

Let $\psi \in \mathbb{C}$ be a solution of the NLS and $\varphi \in \mathbb{C}^2$ be solution of the Lax system:

$$\varphi_x = \left(\begin{array}{cc} \lambda & \psi \\ -\bar{\psi} & -\lambda \end{array} \right) \varphi, \qquad \varphi_t = i \left(\begin{array}{cc} \lambda^2 + \frac{1}{2}|\psi|^2 & \frac{1}{2}\psi_x + \lambda\psi \\ \frac{1}{2}\bar{\psi}_x - \lambda\bar{\psi} & -\lambda^2 - \frac{1}{2}|\psi|^2 \end{array} \right) \varphi.$$

Let $\varphi_1 = (p_1, q_1)$ be a particular (nonzero) solution of the Lax system for $\lambda = \lambda_1 \in \mathbb{C}$. The following one-fold Darboux transformation:

$$\hat{\psi} = \psi + \frac{2(\lambda_1 + \bar{\lambda}_1) p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2},$$

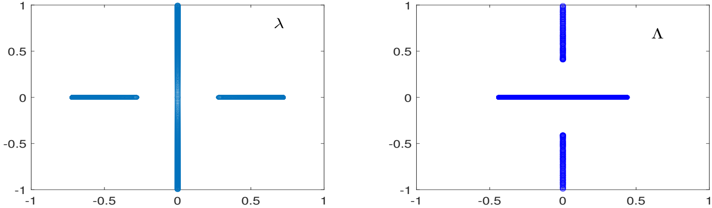
provides a new solution $\hat{\psi}$ of the same NLS equation. Hence, we need

- to know the Lax spectrum (admissible values of λ) for periodic waves,
- to know how to choose λ_1 relative to the Lax spectrum,
- to get explicit expressions for eigenfunction φ_1,
- and to perform the Darboux transformation.

Dnoidal periodic wave

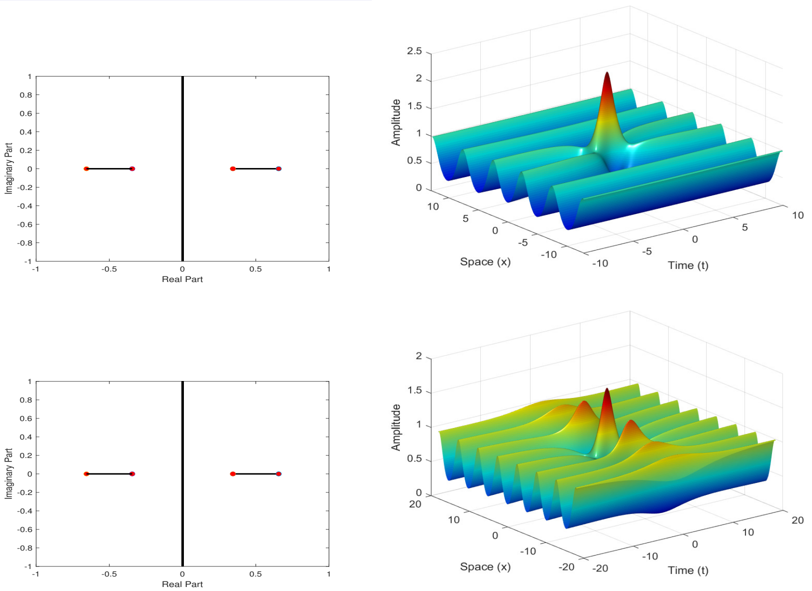
Lax spectrum and modulation stability were studied by Deconinck–Segal (2017). Construction of rogue waves in relation to their modulation stability was in Chen–P (2018), Chen–P–White (2020), Feng–Ling–Takahashi (2020).

For the dnoidal periodic wave $\psi_{\rm dn}(x,t) = {\rm dn}(x;k)e^{i(1-k^2/2)t}$, the Lax spectrum (left) and stability spectrum (right) are:



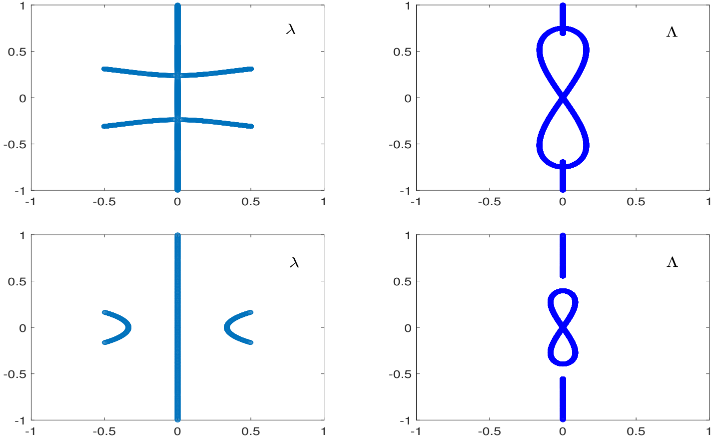
Peregrine soliton corresponds to the end points of λ for which $\Lambda = 0$.
Akhmediev breather corresponds to the interior points of λ for which $\Lambda > 0$.

Two Peregrine solitons on the dnoidal periodic wave

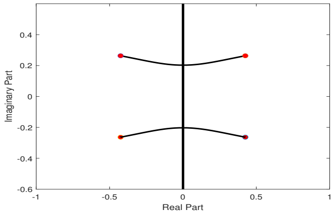


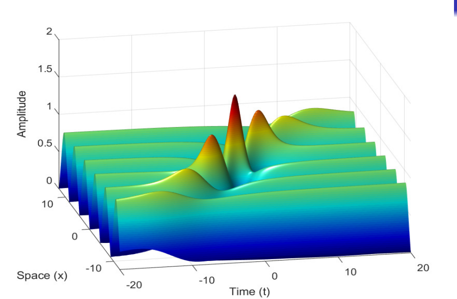
Cnoidal periodic wave

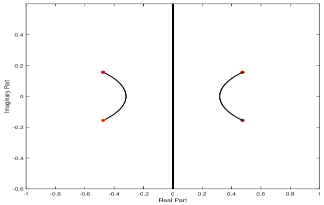
For the cnoidal periodic wave $\psi_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{i(k^2-1/2)t}$, the Lax spectrum (left) and stability spectrum (right) are:

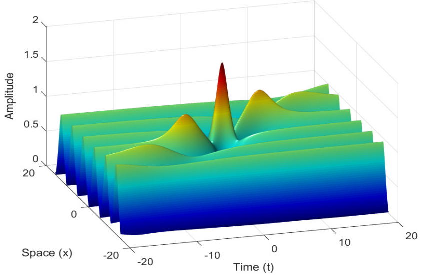


One Peregrine soliton on the cnoidal periodic wave



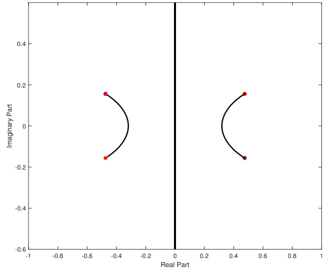


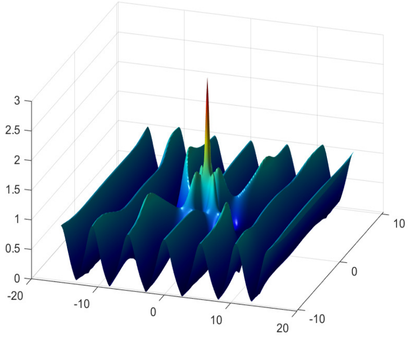




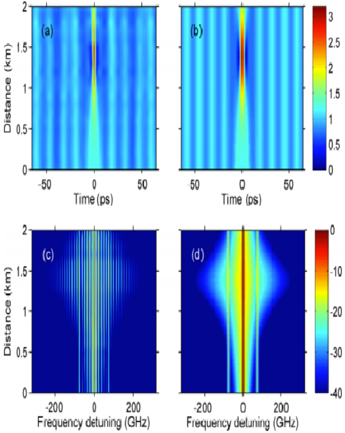
Second-order soliton on the cnoidal periodic wave

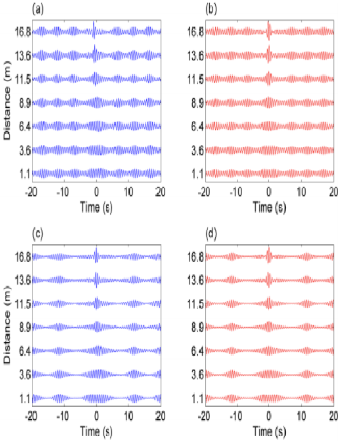
With the two-fold Darboux transformations, one can use both eigenvalues and construct a symmetric rogue wave on the cnoidal background.





The same rogue waves have been observed in optics (left) and hydrodynamics (right): Xu–Chabchoub–P–Kibler (2020).



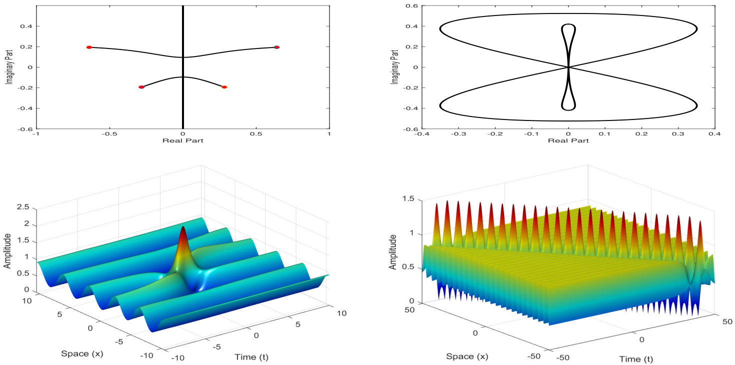


Relation to modulation instability of the periodic wave

For the periodic waves with the nontrivial phase

$$\psi(x,t) = \sqrt{\beta - k^2 \mathrm{sn}^2(x;k)}\, e^{i\Theta(x)} e^{2ibt}, \quad \Theta(x) = -2\int_0^x \frac{dx}{\beta - k^2 \mathrm{sn}^2(x;k)},$$

we can obtain examples of the relation between localization and instability.

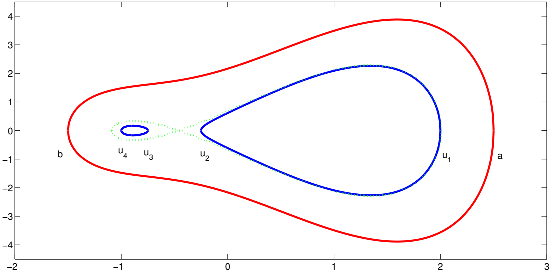


A similar phenomenon for the modified KdV equation

The modified Korteweg–de Vries (mKdV) equation

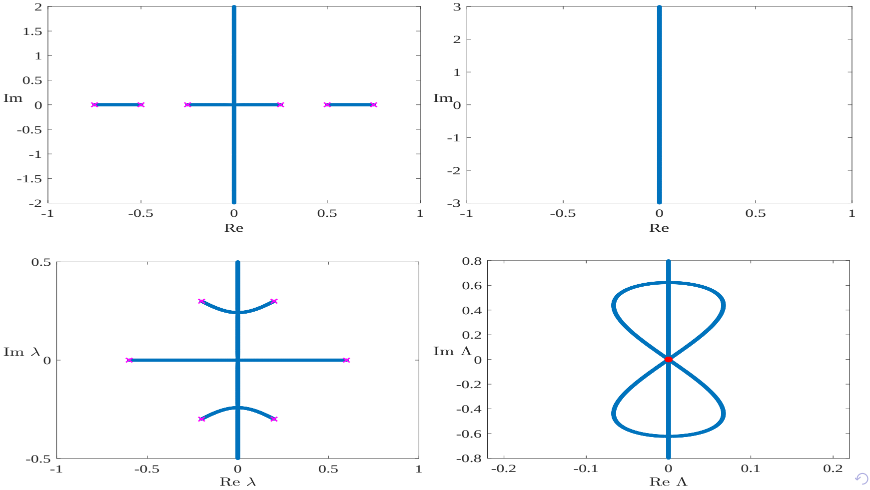
$$u_t + 6u^2 u_x + u_{xxx} = 0$$

admits two families of traveling periodic waves $u(x,t) = \phi(x - ct)$ which generalize the dnoidal and cnoidal waves.

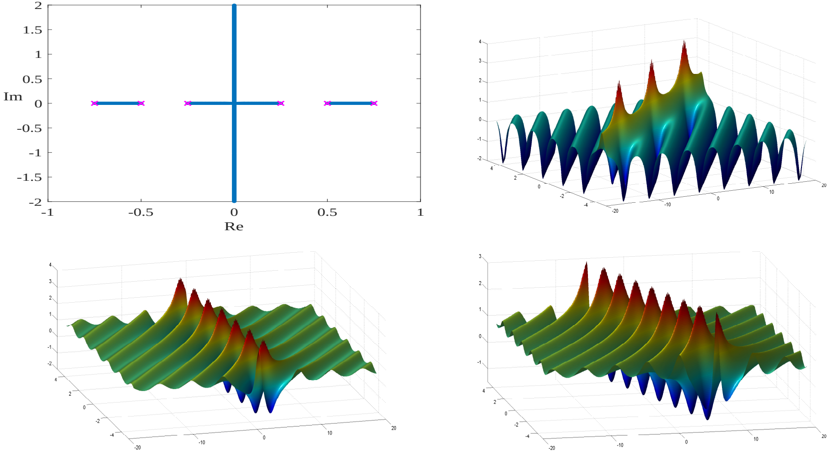


Modulational stability of traveling periodic waves

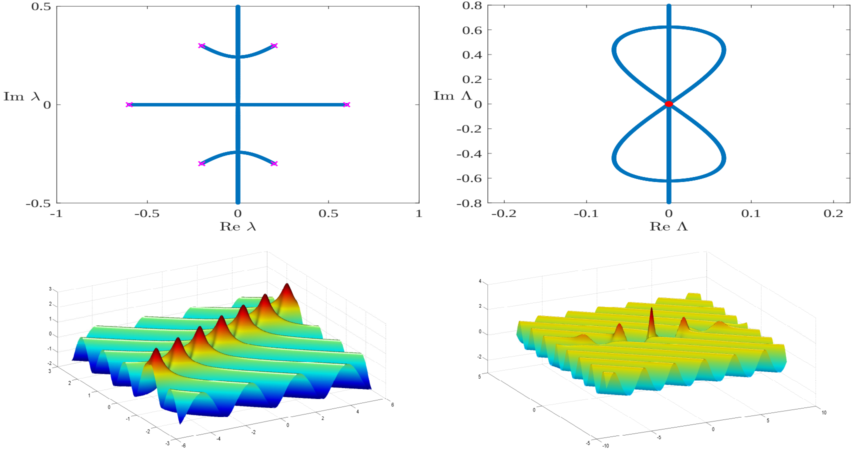
Dnoidal waves are modulationally stable and cnoidal waves are modulationally unstable for all parameter configurations: Bronski–Johnson–Kapitula (2011); Chen–P. (2019); Cui–P (2025)



Modulationally stable dnoidal waves



Modulationally unstable cnoidal waves

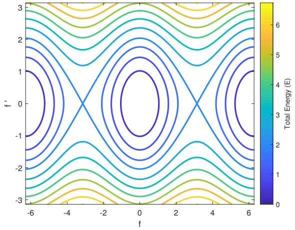


A similar phenomenon for the sine–Gordon equation

The sine–Gordon equation is

$$u_{tt} - u_{xx} + \sin(u) = 0.$$

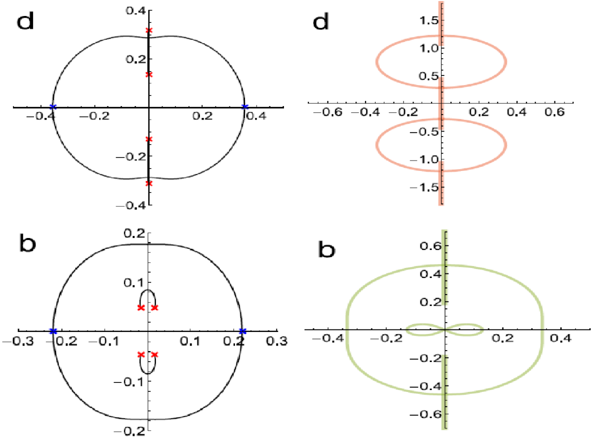
The travelling wave solutions $u(x,t) = f(x - ct)$ with $c > 1$ satisfy (after Lorentz transformation) the differential equation $f'' + \sin(f) = 0$.



Rotational solutions:
$f'(x) = \pm 2k^{-1}\mathrm{dn}(k^{-1}x, k)$.

Librational solutions:
$f'(x) = 2k\mathrm{cn}(x, k)$.

Spectral and modulational instability of periodic waves

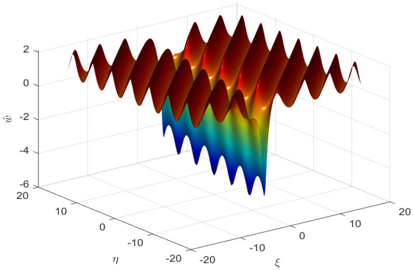


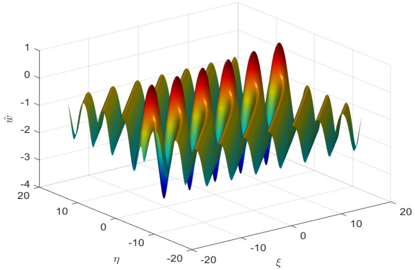
Top:
Rotational waves
Bottom:
Librational waves

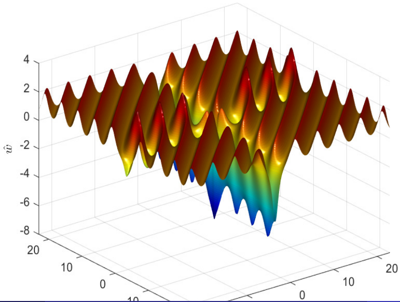
Left:
Lax spectrum
Right:
Stability spectrum

B. Deconinck–P. McGill–B.L. Segal (2017)
C. Jones, R. Marangell, P. Miller, R.G. Plaza (2013).

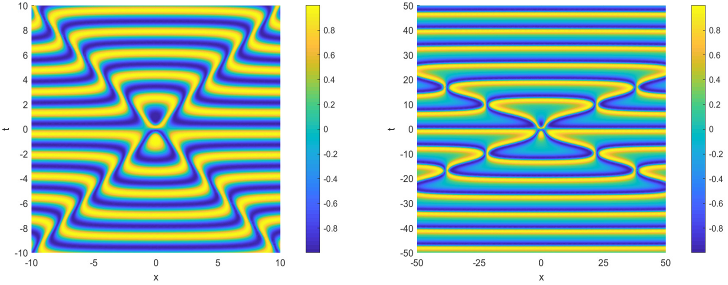
Algebraic solitons on the rotational background



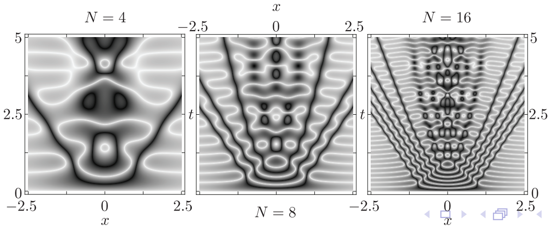




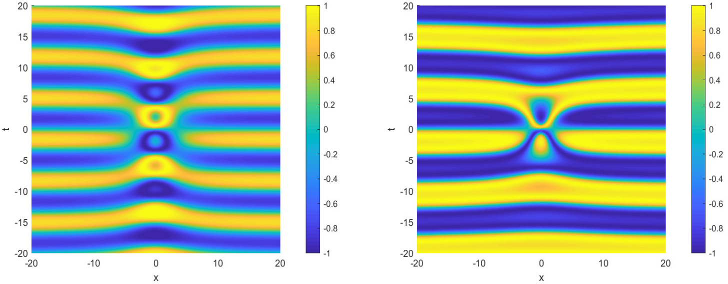
Algebraic solitons for $\sin(u)$ in (x, t) variables



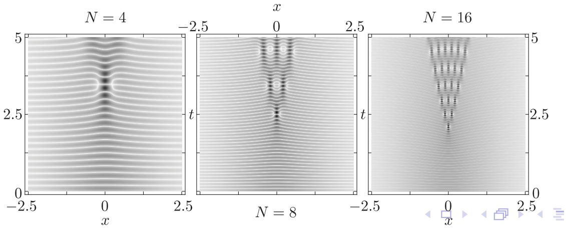
Compare with R.J. Buckingham–P.D. Miller (2013):



Rogue wave on the librational background



Compare with R.J. Buckingham–P.D. Miller (2013):



Legacy of the question from Linz Austria, April 2016

- Derivative NLS equation: Chen–P–Upsal (2021); Chen–P (2021)
- Discrete NLS and mKDV equations: Chen–P (2023); Chen–P (2024)
- Breathers on the stable background (KdV, mKDV): Hoefer–Mucalica–P (2023); Mucalica–P (2024); Arruda–P (2025)
- Nonlocal equations (BO, NLS–BO): Chen–P (2024); Chen–P (2025).

The first two 2018 papers with Jinbing Chen on NLS and mKdV are numbers 13 and 18 in the top list of my cited papers on Google Scholar.

Many thanks for suggesting this question! Heartiest congratulations!!!

Legacy of the question from Linz Austria, April 2016

- Derivative NLS equation: Chen–P–Upsal (2021); Chen–P (2021)
- Discrete NLS and mKDV equations: Chen–P (2023); Chen–P (2024)
- Breathers on the stable background (KdV, mKDV): Hoefer–Mucalica–P (2023); Mucalica–P (2024); Arruda–P (2025)
- Nonlocal equations (BO, NLS–BO): Chen–P (2024); Chen–P (2025).

The first two 2018 papers with Jinbing Chen on NLS and mKdV are numbers 13 and 18 in the top list of my cited papers on Google Scholar.

Many thanks for suggesting this question! Heartiest congratulations!!!

Legacy of the question from Linz Austria, April 2016

- Derivative NLS equation: Chen–P–Upsal (2021); Chen–P (2021)
- Discrete NLS and mKDV equations: Chen–P (2023); Chen–P (2024)
- Breathers on the stable background (KdV, mKDV): Hoefer–Mucalica–P (2023); Mucalica–P (2024); Arruda–P (2025)
- Nonlocal equations (BO, NLS–BO): Chen–P (2024); Chen–P (2025).

The first two 2018 papers with Jinbing Chen on NLS and mKdV are numbers 13 and 18 in the top list of my cited papers on Google Scholar.

Many thanks for suggesting this question! Heartiest congratulations!!!