Stability of smooth travelling waves and instability of peaked travelling waves in the Camassa–Holm equation

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joint work with Anna Geyer (TU Delft) and Fabio Natali (Brazil)

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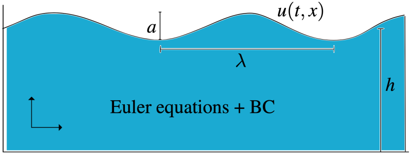
Introduction

The Camassa-Holm equation

$$u_t - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx} \qquad \text{(CH)}$$

models the propagation of unidirectional shallow water waves, where $u = u(t, x)$ represents the water surface. [Camassa & Holm, 1993]



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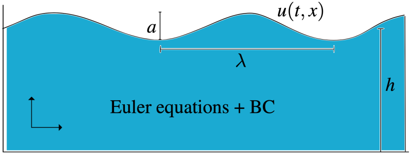

- small amplitude: BBM equation $u_t - u_{txx} + 3\,u\,u_x = 0$
- moderate amplitude: b-family of Camassa-Holm equations
  $u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx} \qquad (b = 2)$

Properties of the Camassa-Holm equation

The local differential equation

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can be rewritten in the integral form of the perturbed Burgers equation

$$u_t + u u_x + \varphi' * \left(u^2 + \frac{1}{2} u_x^2 \right) = 0,$$

where $\varphi := (1 - \partial_x^2)^{-1} \delta_0$ is the Green function.

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The model may feature wave breaking:

$$\|u(t,\cdot)\|_{L^\infty} < \infty, \quad \|u_x(t,\cdot)\|_{L^\infty} \to \infty \quad \text{as} \;\; t \to T < \infty$$

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Solutions of the Burgers equation $v_t + vv_x = 0$ with $v(0,x) = f(x)$ feature the same wave breaking:

$$v(t,x) = f(x - tv(t,x)) \quad \Rightarrow \quad v_x = \frac{f'(x-tv)}{1 + tf'(x-tv)}.$$

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- locally well-posed in H^s, $s > 3/2$ [Constantin & Escher, 1998]
- no continuous dependence in H^s, $s \leq 3/2$
  [Himonas, Grayshan, Holliman (2016)] [Guo, Liu, Molinet, Yin (2018)]
- locally well-posed in $H^1 \cap W^{1,\infty}$.
  [De Lellis, Kappeler, Topalov (2007)] [Linares, Ponce, Sideris (2019)]

Stability of traveling waves

There exist smooth, peaked and cusped periodic waves [Lenells, 2006]

Some previous stability results:

- Orbital stability for peaked periodic and solitary waves in H^1 using variational methods and energy integrals

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- ▷ Stability of cusped waves is open.

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- The traveling wave ϕ is spectrally stable if $\mathcal{L}|_{X_0} \geq 0$.

Standard approach to spectral stability

The standard approach fails for the Camassa–Holm equation

$$u_t - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx} \tag{CH}$$

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- For the smooth periodic waves, the number of negative eigenvalues of $\mathcal{L}$ changes from 1 to 2, depending on parameters.
- For the peaked periodic or solitary waves, the zero eigenvalue of $\mathcal{L}$ is not separated from the continuous spectrum.

The main goal of my talk is to explain how these two problems can be solved on the case study of the Camassa–Holm equation.

Bi-Hamiltonian structure of CH

The CH equation

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has three conserved quantities

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It can be written in Hamiltonian form in two ways:

$$u_t = J F'(u), \quad J = -(1 - \partial_x^2)^{-1}\partial_x$$

and

$$m_t = J_m E'(m), \quad J_m = -\left(m\partial_x + \partial_x m\right),$$

where $m = u - u_{xx}$.

Traveling waves

Smooth traveling waves of the form $u(x,t) = \phi(x - ct)$ satisfy

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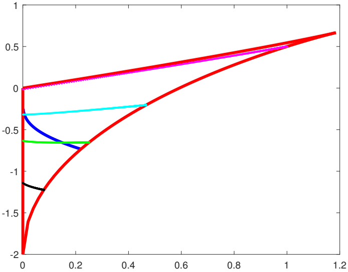
Both second-order equations are compatible iff

$$b = \frac{1}{2}(\phi')^2 - \frac{1}{2}\phi^2 + \frac{a}{c-\phi}.$$

Existence of periodic waves on the (a, b) parameter plane



Periodic waves exist inside the region between three boundaries:

- Peaked waves correspond to the left boundary: $a = 0$.
- Solitary waves correspond to the top boundary.
- Constant waves correspond to the right boundary.

Stability via Standard integration

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which is the Euler–Lagrange equation for the action functional:

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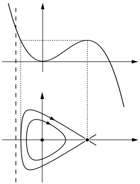
The corresponding linearized operator is $\mathcal{L} : H^2_{\rm per} \subset L^2 \to L^2$,

$$\mathcal{L} = \Lambda''_{c,b}(\phi) = -\partial_x(c-\phi)\partial_x + (c - 3\phi + \phi'').$$

$\sigma(\mathcal{L}) \subset \mathbb{R}$ consists of eigenvalues and $0 \in \sigma(\mathcal{L})$ since $\mathcal{L}\phi' = 0$.

How many negative eigenvalues exist in $\sigma(\mathcal{L})$?

Period function



Fix b and write the second-order equation as the system

$$\begin{cases} x' = y, \\ y' = -\frac{1}{x} V'(x) - \frac{1}{2x} y^2, \end{cases} \qquad \begin{cases} x := c - \phi, \\ y := -\phi', \end{cases}$$

with first integral $H(x,y) := \frac{1}{2} x y^2 + V(x;b)$.
There is a continuum of periodic orbits $\gamma(a)$ in $\{H(x,y) = a\}$ with the period given by the period function

$$\mathfrak{L}(a) = \int_{\gamma(a)} \frac{dx}{y}.$$

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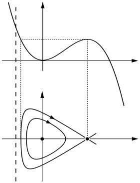

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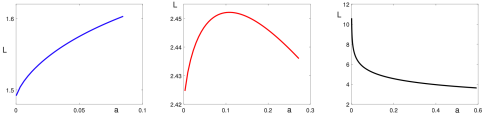
$\mathfrak{L}(a)$ change monotonicity for different b. [Geyer & Villadelprat, 2015]



Negative eigenvalues in $\sigma(\mathcal{L})$

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We have $\mathcal{L}\phi' = 0$ and $\mathcal{L}\partial_a \phi = 0$, with $v = c_1 \phi' + c_2 \partial_a \phi$ being a general solution of $\mathcal{L}v = 0$.

- $\mathfrak{L}'(a) > 0 : \quad \sigma(\mathcal{L}) = \{-\lambda_1, -\lambda_2, 0, \dots\}$
- $\mathfrak{L}'(a) = 0 : \quad \sigma(\mathcal{L}) = \{-\lambda_1, 0, 0, \dots\}$
- $\mathfrak{L}'(a) < 0 : \quad \sigma(\mathcal{L}) = \{-\lambda_1, 0, \dots\}$

[M. Johnson, 2009] [A. Neves, 2009]

Standard approach to spectral stability is computationally hard.

Stability via Alternative integration

Alternative integration, after multiplication by $(c-\phi)$, gives

$$-(c-\phi)^2(\phi''-\phi) = a, \quad a \in \mathbb{R}.$$

which can be written as $(c-\phi)^3\mu = a(c-\phi)$ for $\mu := \phi - \phi''$.

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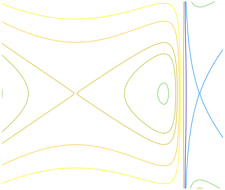
The corresponding linearized operator is $\mathcal{K} : L^2_{\rm per} \to L^2_{\rm per}$,

$$\mathcal{K} := (c-\phi)^3 - 2a(1-\partial_x^2)^{-1}, \qquad \mathcal{K}\mu' = 0.$$

$\sigma(\mathcal{K}) \subset \mathbb{R}$ consists of eigenvalues below $\min_{x\in[0,L]}(c-\phi)^3 > 0$ and the continuous spectrum in $[\min_{x\in[0,L]}(c-\phi)^3, \max_{x\in[0,L]}(c-\phi)^3]$.

How many negative eigenvalues exist in $\sigma(\mathcal{K})$?

Period function



Fix a and write the second-order equation as

$$\begin{cases} x' = y, \\ y' = x + \frac{a}{(c-x)^2}, \end{cases} \qquad \begin{cases} x := \phi, \\ y := \phi', \end{cases}$$

with Hamiltonian $H(x,y) = \frac{1}{2}y^2 + V(x;a)$.
There is a continuum of periodic orbits $\gamma(b)$ in $\{H(x,y) = b\}$.

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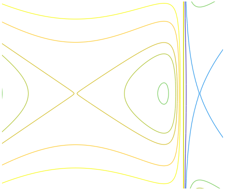

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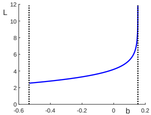
The period function is defined as

$$\mathfrak{L}(b) = \int_{\gamma(b)} \frac{dx}{y}.$$



$\mathfrak{L}'(b) > 0$ for all values of a.
[Geyer, Martins, Natali, & Pelinovsky, 2022]

Negative eigenvalues in $\sigma(\mathcal{K})$

What does it imply for the linearized operator $\mathcal{K}$?

$$\mathcal{K} := (c - \phi)^3 - 2a(1 - \partial_x^2)^{-1}.$$

We have $\mathcal{K}\mu' = 0$ and $\mathcal{K}\partial_b\mu = 0$, where $\mu := \phi - \phi''$. Hence, $v = c_1\mu' + c_2\partial_b\mu$ is a general solution of $\mathcal{K}v = 0$.

For the negative spectrum of $\mathcal{K}$ we find

- $\mathfrak{L}'(b) < 0 : \quad \sigma(\mathcal{K}) = \{-\lambda_1, -\lambda_2, 0, \dots\}$
- $\mathfrak{L}'(b) = 0 : \quad \sigma(\mathcal{K}) = \{-\lambda_1, 0, 0, \dots\}$
- $\mathfrak{L}'(b) > 0 : \quad \sigma(\mathcal{K}) = \{-\lambda_1, 0, \dots\}$

Since $\mathfrak{L}'(b) > 0$, $\sigma(\mathcal{K})$ admits only one simple negative eigenvalue.

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Constrained minimizers of energy

Recall the three conserved quantities

$$M(u) = \int_0^L u dx, \quad E(u) = \frac{1}{2}\int_0^L (u^2 + u_x^2) dx, \quad F(u) = \frac{1}{2}\int_0^L (u^3 + u u_x^2)\, dx,$$

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The constrained space is

$$X_0 := \left\{ u \in L^2_{\rm per} : \quad \langle 1, u \rangle = 0, \quad \langle \phi - \phi'', u \rangle = 0 \right\}.$$

In variable $m := u - u_{xx}$ for $u \in H^2_{\rm per}$, the constraints become

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How many negative eigenvalues exist in $\sigma(\mathcal{K}|_{Y_0})$?

Constrained minimizers of energy - scalar case

Consider the constrained spectral problem for $K|_{Y_0}$:

$$\mathcal{K}m = \lambda m - \alpha\phi, \quad \langle \phi, m \rangle = 0,$$

where α is a Lagrange multiplier.

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and hence $\langle \phi, m \rangle = \underbrace{\langle (\mathcal{K} - \lambda I)^{-1}\phi, \phi \rangle}_{=: A(\lambda)} = 0.$

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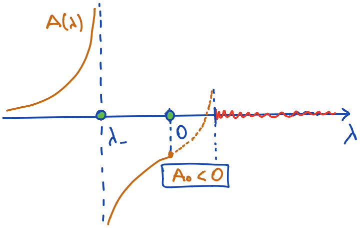

If $\lim_{\lambda \uparrow 0} A(\lambda) = \langle \mathcal{K}^{-1}\phi, \phi \rangle < 0$,
then $\mathcal{K}|_{Y_0} \geq 0$.

[Vakhitov–Kolokolov, 1974]

Constrained minimizers of energy - vector case

Theorem

Let $A(\lambda)$ be the matrix-valued function defined by

$$A_{ij}(\lambda) := \langle (\mathcal{K} - \lambda I)^{-1} v_i, v_j \rangle, \quad 1 \leq i,j \leq N, \quad \lambda \notin \sigma(\mathcal{K}),$$

where $\langle p, v_i \rangle = 0$ for $p \in Y_0$ and let $A_0 := \lim_{\lambda \uparrow 0} A(\lambda)$. Then,

$$n(\mathcal{K}\big|_{Y_0}) = n(\mathcal{K}) - n_0 - z_0, \quad z(\mathcal{K}\big|_{Y_0}) = z(\mathcal{K}) + 2z_0 + n_0 + p_0 - N,$$

where n_0, z_0, and p_0 are the numbers of negative, zero, and positive eigenvalues of A_0 and $N = \dim Y_0$.

[Pelinovsky, 2011], [Kapitula–Promislow, 2013]

Sharp condition that $\mathcal{K}|_{Y_0} \geq 0$

We find that for $Y_0 = \left\{ m \in L^2_{\rm per} : \langle 1, m \rangle = 0, \langle \phi, m \rangle = 0 \right\}$,

$$A_0 = \begin{bmatrix} \langle \mathcal{K}^{-1} 1, 1 \rangle & \langle \mathcal{K}^{-1} \phi, 1 \rangle \\ \langle \mathcal{K}^{-1} 1, \phi \rangle & \langle \mathcal{K}^{-1} \phi, \phi \rangle \end{bmatrix} = \begin{bmatrix} -\frac{1}{2a} \partial_c M & -\partial_a M - \frac{c}{2a} \partial_c M \\ -\frac{1}{2a} \partial_c E & -\partial_a E - \frac{c}{2a} \partial_c E \end{bmatrix},$$

where E and M are momentum and mass functionals as function of (a, c) along the fixed period curve $\mathfrak{L}(a, b, c) = L$.

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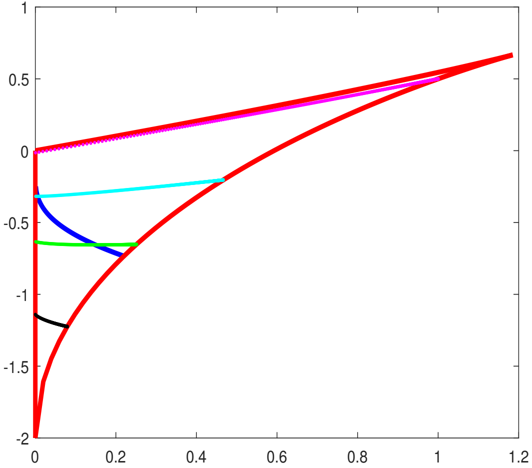
where E and M are momentum and mass functionals as function of (a, c) along the fixed period curve $\mathfrak{L}(a, b, c) = L$.

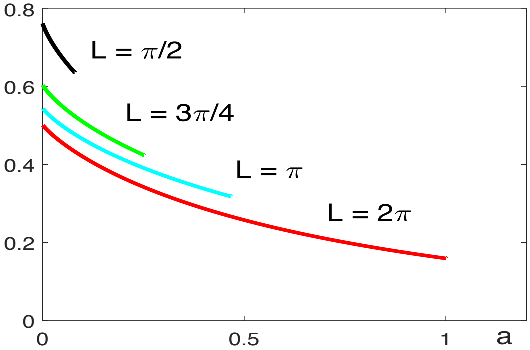
Since $n(\mathcal{K}) = 1$, then $\mathcal{K}|_{Y_0} \geq 0$ if and only if

$$\det(A_0) = \frac{1}{2a} \left[\partial_c M \partial_a E - \partial_a M \partial_c E \right] \leq 0 \iff \frac{d}{da} \left(\frac{E}{M^2} \right) \leq 0.$$

Sharp condition $\frac{d}{da}\left(\frac{E}{M^2}\right) \leq 0$ for $\mathcal{K}|_{Y_0} \geq 0$



Sharp condition $\frac{d}{da}\left(\frac{E}{M^2}\right) \leq 0$ for $\mathcal{K}|_{Y_0} \geq 0$

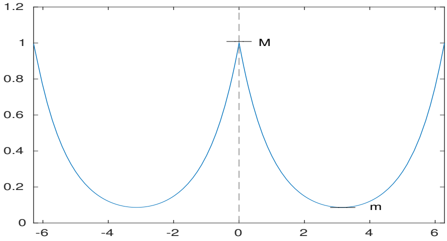


Existence of peaked periodic waves

Let $\varphi(x)$ be the Green function satisfying $(1-\partial_x^2)\varphi = \delta$ such that the CH equation is written as

$$\begin{cases} u_t + uu_x + \varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0, \\ u|_{t=0} = u_0. \end{cases}$$

Green function gives the peaked TW $u(x,t) = \varphi(x - ct)$ with $c = \varphi(0)$ so that $c - \varphi \geq 0$. Hence, $\mathcal{K} := (c-\varphi)^3 - 2a(1-\partial_x^2)^{-1}$ does not have spectral gap near zero eigenvalue.



Stability of peaked periodic waves

Theorem (Constantin–Molinet (2001); Lenells (2005))

φ is a unique (up to translation) minimizer of $F(u)$ in H^1 subject to $E(u)$ and $M(u)$.

Theorem (Constantin–Strauss (2000); Lenells (2005))

For every small $\varepsilon > 0$, if the initial data satisfies

$$\|u_0 - \varphi\|_{H^1} < \left(\frac{\varepsilon}{3}\right)^4,$$

then the solution satisfies

$$\|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0,T),$$

where $\xi(t)$ is a point of maximum for $u(t,\cdot)$.

Instability of peakons

Consider solutions of the Cauchy problem:

$$\left\{ \begin{array}{l} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{array} \right. \qquad Q[u] := \varphi' * \left(u^2 + \frac{1}{2} u_x^2 \right).$$

Assume that u_0 is piecewise C^1 with a single peak.

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Assume that u_0 is piecewise C^1 with a single peak.

Theorem (Natali–P. (2020); Madiyeva–P (2021))

For every $\delta > 0$*, there exist* $t_0 > 0$ *and* $u_0 \in H^1 \cap W^{1,\infty}$ *satisfying*

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

s.t. the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ *with* $T > t_0$ *satisfies*

$$\|u_x(t_0, \cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ *is a point of peak of* $u(t, \cdot)$ *for* $t \in [0, T)$.

Instability of peakons

Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{cases} \qquad Q[u] := \varphi' * \left(u^2 + \frac{1}{2} u_x^2 \right).$$

Assume that u_0 is piecewise C^1 with a single peak.

Weak formulation of the unique global conservative solution:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\psi_t + \frac{1}{2} u^2 \psi_x - Q[u]\psi \right) dxdt + \int_{\mathbb{R}} u_0(x)\psi(0,x)dx = 0,$$

where $\psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$.

Instability of peakons

Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{cases} \qquad Q[u] := \varphi' * \left(u^2 + \frac{1}{2}u_x^2\right).$$

Assume that u_0 is piecewise C^1 with a single peak.

- ▷ If $u \in H^1(\mathbb{R})$, then $Q[u] \in C(\mathbb{R})$.
- ▷ If $u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, then $Q[u]$ is Lipschitz continuous.

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Assume that u_0 is piecewise C^1 with a single peak.

If $u(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$ for $t \in [0,T)$. Then, $\xi(t) \in C^1(0,T)$ and

$$\frac{d\xi}{dt} = u(t,\xi(t)), \quad t \in (0,T).$$

Decomposition near a single peakon

Consider a decomposition:

$$u(t,x) = \varphi(x - ct - a(t)) + v(t, x - ct - a(t)), \quad t \in [0,T), \quad x \in \mathbb{R},$$

with the peak at $\xi(t) = ct + a(t)$ for $v(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$.

Then,

$$(\varphi - c)\varphi' + Q(\varphi) = 0,$$
$$\frac{da}{dt} = v(t,0),$$

and

$$v_t = (c-\varphi)v_x + (v|_{x=0} - v)\varphi' + (v|_{x=0} - v)v_x - \varphi' * (\varphi v + \frac{1}{2}\varphi' v_x) - Q[v].$$

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Due to

$$[v(0) - v(x)]\varphi'(x) - \varphi' * \varphi v - \frac{1}{2}\varphi' * \varphi' v_x = \varphi(x) \int_0^x v(y)dy,$$

the evolution of $v(t,x)$ simplifies to

$$v_t = (c - \varphi)v_x + \varphi w + (v|_{x=0} - v)v_x - Q[v],$$

where $w(t,x) = \int_0^x v(t,y)dy$.

Linearized evolution

Truncation of the quadratic terms yields the linearized problem:

$$\begin{cases} v_t = (c - \varphi)v_x + \varphi w, \quad t > 0, \\ v|_{t=0} = v_0(x), \end{cases}$$

where $w(t,x) = \int_0^x v(t,y)dy$.

Solution with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s), \quad w(t, X(t,s)) = W(t,s).$$

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The evolution problem splits into

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - c, \\ X|_{t=0} = s, \end{cases} \qquad \begin{cases} \frac{dW}{dt} = \varphi'(X) W, \\ W|_{t=0} = w_0(s), \end{cases} \qquad \begin{cases} \frac{dV}{dt} = \varphi(X) W, \\ V|_{t=0} = v_0(s). \end{cases}$$

Since φ is Lipschitz, there exists unique characteristic function $X(t,s)$ for each $s \in \mathbb{R}$. The peak location $X(t,0) = 0$ is invariant in time.

Properties of the linearized evolution

Assume $v_0 \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$. For every $t > 0$, we proved that

- $\|v(t,\cdot)\|_{L^\infty} \leq C$ for some $C > 0$.
- $\|v(t,\cdot)\|^2_{H^1} = C_+ e^t + C_0 + C_- e^{-t}$ for some C_+, C_0, C_-.

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- $\|v(t,\cdot)\|^2_{H^1} = C_+ e^t + C_0 + C_- e^{-t}$ for some C_+, C_0, C_-.

It may seem that the growth of $\|v(t,\cdot)\|^2_{H^1}$ contradicts to H^1 orbital stability of peakons, but $v(t,\cdot)$ satisfies the linearized equations of motion and indicates linear and spectral instability of peakons in H^1.

Illustration of the linear instability

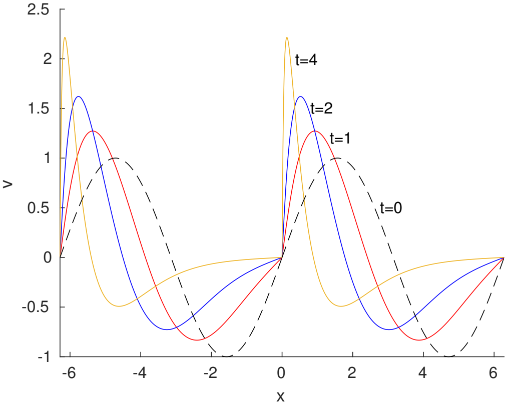

Figure: The plots of $v(t,x)$ versus x on $[-2\pi, 2\pi]$ for different values of t in the case $v_0(x) = \sin(x)$.

Nonlinear evolution

Recall the evolution problem:

$$\begin{cases} v_t = (c - \varphi)v_x + \varphi w + (v|_{x=0} - v)v_x - Q[v], \quad t \in (0, T), \\ v|_{t=0} = v_0(x), \end{cases}$$

where $w(t,x) = \int_0^x v(t,y)dy$ and $Q[v] := \varphi' * \left(v^2 + \frac{1}{2}v_x^2\right)$.

Solution with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s), \quad w(t, X(t,s)) = W(t,s).$$

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where $w(t,x) = \int_0^x v(t,y)dy$ and $Q[v] := \varphi' * \left(v^2 + \frac{1}{2}v_x^2\right)$.

The characteristic coordinates $X(t,s)$ satisfies

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1 + v(t,X) - v(t,0), & t \in (0,T), \\ X|_{t=0} = s. \end{cases}$$

Since φ is Lipschitz, there exists the unique characteristic function $X(t,s)$ for each $s \in \mathbb{R}$ if $v(t,\cdot)$ remains in $H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$
The peak location $X(t,0) = 0$ is invariant in time.

Instability theorem

Theorem (Natali–P. (2020); Madiyeva–P (2021))

For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

such that the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ with $T > t_0$ satisfies

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ is a point of peak of $u(t,\cdot)$ for $t \in [0,T)$.

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$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ is a point of peak of $u(t,\cdot)$ for $t \in [0,T)$.

From the right side of the peak, $V_0(t) = v(t,0)$, $U_0(t) = v_x(t,0^+)$:

$$\frac{dU_0}{dt} = U_0 + V_0 + V_0^2 - \frac{1}{2}U_0^2 - P[v](0), \quad P[v] := \varphi * \left(v^2 + \frac{1}{2}v_x^2\right).$$

Proof of instability

From orbital stability in H^1 [A. Constant, W. Strauss (2000)]
If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

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From the equation on the right side of the peak:

$$\frac{dU_0}{dt} = U_0 + V_0 + V_0^2 - \frac{1}{2}U_0^2 - P[v](0)$$

and since $P[v] > 0$, we have

$$\frac{dU_0}{dt} \leq U_0 + C\varepsilon \quad \Rightarrow \quad U_0(t) \leq [U_0(0) + C\varepsilon]\, e^t$$

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$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

If $U_0(0) = -2C\varepsilon$, then

$$U_0(t) \leq -C\varepsilon e^t,$$

hence $|U_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

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hence $|U_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

The initial constraint $\|v_0\|_{L^\infty} + \|v_0'\|_{L^\infty} < \delta$, is satisfied
if $\forall\delta > 0$, $\exists\varepsilon > 0$ such that

$$\left(\frac{\varepsilon}{3}\right)^4 + 2C\varepsilon < \delta.$$

Strong instability theorem

Theorem (Natali–P. (2020); Madiyeva–P (2021))

For every $\delta > 0$, there exist $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

such that the maximal existence time of the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ is finite.

Strong instability theorem

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For every $\delta > 0$, there exist $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

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From the right side of the peak, $V_0(t) = V(t,0)$, $U_0(t) = U(t,+0)$:

$$\frac{dU_0}{dt} = U_0 + V_0 + V_0^2 - \frac{1}{2}U_0^2 - P[v](0) \leq U_0 - \frac{1}{2}U_0^2 + C\varepsilon.$$

is controlled by Ricatti differential inequality.

Strong instability theorem

Theorem (Natali–P. (2020); Madiyeva–P (2021))

For every $\delta > 0$, there exist $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

such that the maximal existence time of the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ is finite.

By the ODE comparison theory, $U_0(t) \leq \overline{U}(t)$, where the supersolution satisfies

$$\frac{d\overline{U}}{dt} = \overline{U} - \frac{1}{2}\overline{U}^2 + C\varepsilon$$

with $U_0(0) = \overline{U}(0) = -C\varepsilon$ and $\overline{U}(t) \to -\infty$ as $t \to \overline{T}$.

Concluding remarks

1. Instability of peakons with respect to peaked perturbations is consistent with local well-posedness for $u_0 \in H^1 \cap W^{1,\infty}$ and wave breaking in a finite time: $u_x(t,x) \to -\infty$ at some $x \in \mathbb{R}$.
   [De Lellis, Kappeler & Topalov (2007)] [Linares, Ponce, & Sideris (2019)]

2. It follows from the method of characteristics that if $v_0 \in C^1(\mathbb{R})$, then $v(t,\cdot) \notin C^1(\mathbb{R})$ for $t > 0$ due to the single peak at $x = \xi(t)$:

$$u(t,x) = \varphi(x - ct - a(t)) + v(t, x - ct - a(t)), \quad t \in [0,T).$$

3. The H^1 orbital stability results on peakons are misleading as the perturbations near the peakon are growing in $W^{1,\infty}$ norm.

4. Instability of peakons can be confirmed from the spectral stability analysis for the b-family of Camassa-Holm equations
   [Lafortune & Pelinovsky (2022)]

Summary

We considered the Camassa–Holm equation

$$u_t + 3uu_x - u_{txx} = 2u_x u_{xx} + uu_{xxx}.$$

which models small-amplitude waves in shallow fluids.

- Smooth periodic and solitary waves are stable in $H^1 \cap W^{1,\infty}$
  - Key idea: use alternative Hamiltonian structure
  - Linearized operator has only one negative eigenvalue
  - TW is constrained minimizer of action functional
- Peaked periodic and solitary waves are unstable in $H^1 \cap W^{1,\infty}$
  - LWP only holds in $H^1 \cap W^{1,\infty}$.
  - Perturbations are bounded in H^1.
  - Perturbations grow in $W^{1,\infty}$.