Nonlinear waves on periodic graphs

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Background: periodic potentials

In many problems (BECs, photonics, optics), wave dynamics is modeled with the cubic nonlinear Schrödinger (Gross–Pitaevskii) equation with a periodic potential

$$iu_t = -u_{xx} + V(x)u \pm |u|^2 u,$$

where $V(x) = V(x+L)$ is bounded and the two different signs correspond to either defocusing (repelling) or focusing (attractive) nonlinearity.

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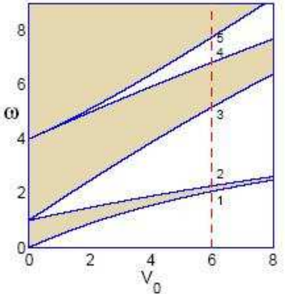
Stationary solutions $u(x,t) = \phi(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a stationary Schrödinger equation with a periodic potential

$$\omega\phi = -\phi_{xx} + V(x)\phi \pm |\phi|^2\phi$$

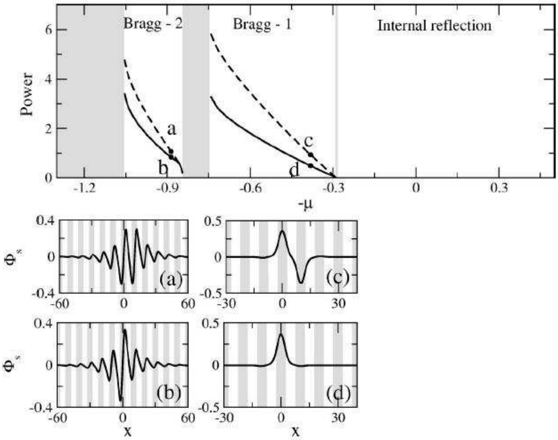
Spectrum of $L = -\partial_x^2 + V_0 \sin^2(x)$:



J. Yang; M. Weinstein;
T. Dohnal; G. Schneider;
V. Konotop; G. Alfimov;

Background: gap solitons

For the defocusing case, the bifurcation diagram is

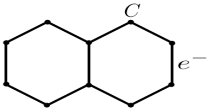


The bifurcation diagram can be understood with the effective NLS equation:

$$\Omega A = -\frac{1}{2}\partial_\ell^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A \pm \nu |A|^2 A, \quad \nu > 0.$$

Graph models

Graph models for the dynamics of constrained quantum particles were first suggested by Pauling and then used by Ruedenberg and Scherr in 1953 to study the spectrum of aromatic hydrocarbons.

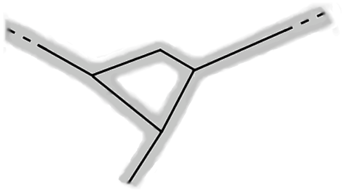


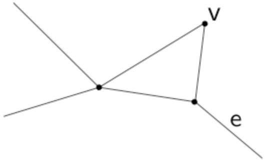
Nowadays graph models are widely used in the modeling of quantum dynamics of thin graph-like structures (quantum wires, nanotechnology, large molecules, periodic arrays in solids, photonic crystals...).

- G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs* (AMS, Providence, 2013).
- P. Exner and H. Kovarík, *Quantum Waveguides*, (Springer, Heidelberg, 2015).

Metric Graphs

Graphs are one-dimensional approximations for constrained dynamics in which transverse dimensions are small with respect to longitudinal ones.



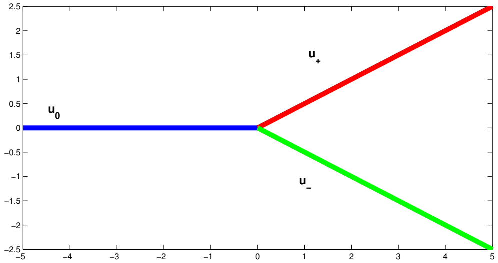


A metric graph Γ is given by a set of edges and vertices, with a metric structure on each edge. Proper boundary conditions are needed on the vertices to ensure that certain differential operators defined on graphs are self-adjoint.

Kirchhoff boundary conditions:

- Functions in each edge have the same value at each vertex.
- Sum of fluxes (signed derivatives of functions) is zero at each vertex.

Example: Y junction graph



The Laplacian operator on the graph Γ is defined by

$$\Delta\Psi = \left[\begin{array}{ll} u_0''(x), & x \in (-\infty, 0), \\ u_\pm''(x), & x \in (0, \infty) \end{array} \right],$$

acting on functions in the form

$$\Psi = \left[\begin{array}{ll} u_0(x), & x \in (-\infty, 0) \\ u_\pm(x), & x \in (0, \infty) \end{array} \right],$$

in the domain

$$\mathcal{D}(\Gamma) = \left\{ \begin{array}{l} (u_0, u_+, u_-) \in H^2(\mathbb{R}^-) \times H^2(\mathbb{R}^+) \times H^2(\mathbb{R}^+) : \\ u_0(0) = u_+(0) = u_-(0), \quad u_0'(0) = u_+'(0) + u_-'(0) \end{array} \right\}.$$

Laplacian on the Y junction graph

Lemma

The operator $\Delta : \mathcal{D}(\Gamma) \to L^2(\Gamma)$ *is self-adjoint.*

The Kirchhoff boundary conditions are symmetric:

$$\begin{aligned}\langle \Phi, \Delta\Psi \rangle - \langle \Delta\Phi, \Psi \rangle &= \left[\bar{v}_0' u_0 - \bar{v}_0 u_0'\right]_{x=0} - \left[\bar{v}_+' u_+ - \bar{v}_+ u_+'\right]_{x=0} - \left[\bar{v}_-' u_- - \bar{v}_- u_-'\right]_{x=0} \\ &= 0,\end{aligned}$$

where $\Phi = (v_0, v_+, v_-)$ and $\Psi = (u_0, u_+, u_-)$ satisfy the Kirchhoff conditions:

$$\left\{\begin{array}{l} u_0(0) = u_+(0) = u_-(0), \\ u_0'(0) = u_+'(0) + u_-'(0). \end{array}\right.$$

Moreover, Δ is self-adjoint under generalized Kirchhoff boundary conditions

$$\left\{\begin{array}{l} \alpha_0 u_0(0) = \alpha_+ u_+(0) = \alpha_- u_-(0) \\ \alpha_0^{-1} u_0'(0) = \alpha_+^{-1} u_+'(0) + \alpha_-^{-1} u_-'(0), \end{array}\right.$$

where $\alpha_0, \alpha_+, \alpha_-$ are arbitrary nonzero parameters.

NLS on the Y junction graph

So far, $\alpha_0, \alpha_+, \alpha_-$ are arbitrary. Let us connect these parameters with the nonlinear coefficients of a nonlinear Schrödinger equation defined on the graph Γ:

$$i\partial_t u_0 + \partial_x^2 u_0 + \alpha_0^2 |u_0|^2 u_0 = 0, \quad x < 0,$$
$$i\partial_t u_\pm + \partial_x^2 u_\pm + \alpha_\pm^2 |u_\pm|^2 u_\pm = 0, \quad x > 0,$$

subject to the generalized Kirchhoff boundary conditions at $x = 0$.

The charge (power) functional

$$Q = \int_{-\infty}^{0} |u_0|^2 dx + \int_0^{+\infty} |u_+|^2 dx + \int_0^{+\infty} |u_-|^2 dx$$

is constant in time t (related to the gauge symmetry).

The Hamiltonian (energy) functional

$$E = \int_{-\infty}^{0} \left(|\partial_x u_0|^2 - \frac{\alpha_0^2}{2} |u_0|^4 \right) dx + \text{similar terms for } u_\pm,$$

is constant in time t (related to the time translation symmetry).

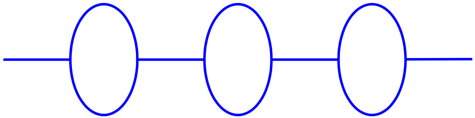
NLS equation on star graphs

- Scattering and stability of solitary waves by R. Adami; C. Cacciapuoti; D. Finco; D.Noja (2011-2014).
- Existence and non-existence of ground states on unbounded graphs by R. Adami; E. Serra; P. Tilli (2014-2016).
- Understanding Kirchhoff boundary conditions in the limit of thin graphs by Z. Sobirov; H. Uecker (2014-2016).
- Reflectionless transmission of solitary waves on the graph vortices by D. Matrasulov; K. Sabirov; D. Dytukh; J.G. Caputo; (2014-2016).
- Dynamical system methods for existence, bifurcations, and stability on tadpole, dumbbell, and periodic graphs.

References:

D.Noja, D.P., and G.Shaikhova, Nonlinearity **28** (2015), 2343;
J. Marzuola and D.P., Applied Math. Research Express **2016**, 98–145;
D.P. and G. Schneider, arXiv:1603.05463 (2016);
S. Gilg, D.P., and G. Schneider, (2016).

Periodic Graph



Let the periodic graph Γ consist of the circles of the normalized length 2π and the horizontal links of the length L. Writing the periodic graph as

$$\Gamma = \oplus_{n\in\mathbb{Z}}\Gamma_n, \qquad \text{with} \qquad \Gamma_n = \Gamma_{n,0} \oplus \Gamma_{n,+} \oplus \Gamma_{n,-},$$

we parameterize $\Gamma_{n,0} := [nP, nP + L]$ and $\Gamma_{n,\pm} := [nP + L, (n+1)P]$, where $P = L + \pi$ is the graph period.

The NLS equation on the periodic graph Γ,

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0, \quad t \in \mathbb{R}, \quad x \in \Gamma, \qquad (1)$$

subject to the Kirchhoff boundary conditions at the vertices.

Linear spectral problem

The spectral problem with a bounded 2π-periodic potential V,

$$\lambda w = -\partial_x^2 w, \quad x \in \Gamma,$$

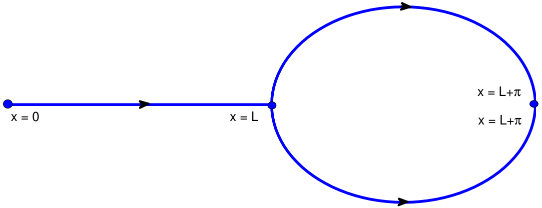
subject to the Kirchhoff boundary conditions for $n \in \mathbb{Z}$,

$$\begin{cases} w_{n,0}(nP+L) = w_{n,+}(nP+L) = w_{n,-}(nP+L), \\ w_{n+1,0}((n+1)P) = w_{n,+}((n+1)P) = w_{n,-}((n+1)P), \end{cases}$$

and

$$\begin{cases} \partial_x w_{n,0}(nP+L) = \partial_x w_{n,+}(nP+L) + \partial_x w_{n,-}(nP+L), \\ \partial_x w_{n+1,0}((n+1)P) = \partial_x w_{n,+}((n+1)P) + \partial_x w_{n,-}((n+1)P). \end{cases}$$

E. Korotyaev and I. Lobanov, Ann. Henri Poincare **8** (2007), 1151
P. Kuchment and O. Post, Commun Math. Phys. **275** (2007), 805



Decomposition of the spectrum on Γ

Lemma

The linear operator $-\partial_x^2 : \mathcal{D}(\Gamma) \to L^2(\Gamma)$ is self-adjoint. Its spectrum $\sigma(-\partial_x^2)$ is positive and consists of two parts.

Integrating by parts with Kirchhoff boundary conditions, we have

$$\lambda \|w\|_{L^2(\Gamma)}^2 = \|\partial_x w\|_{L^2(\Gamma)}^2 \geq 0.$$

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The first part of $\sigma(-\partial_x^2)$ corresponds to the eigenfunctions of the form

$$\begin{cases} w_{n,0}(x) = 0, & x \in [nP, nP+L], \\ w_{n,+}(x) = -w_{n,-}(x), & x \in [nP+L, (n+1)P], \end{cases} \quad n \in \mathbb{Z}.$$

Clearly, $\lambda = m^2$, $m \in \mathbb{N}$ is an eigenvalue of infinite multiplicity with the eigenfunction $w_{n,\pm}(x) = \pm\delta_{n,k} \sin[m(x - 2\pi n)]$, $k \in \mathbb{Z}$.

The second part of $\sigma(-\partial_x^2)$ corresponds to the eigenfunctions of the form

$$w_{n,+}(x) = w_{n,-}(x), \quad x \in [nP+L, (n+1)P], \quad n \in \mathbb{Z}.$$

Construction of symmetric eigenfunctions

Let us parameterize the spectral parameter $\lambda = \omega^2$. Then, solutions of ODEs are found in terms of the boundary conditions:

$$\begin{cases} w_{n,0}(x) = a_n \cos(\omega(x - nP)) + b_n \sin(\omega(x - nP)), & x \in [nP, nP + L], \\ w_{n,\pm}(x) = c_n \cos(\omega(x - nP - L)) + d_n \sin(\omega(x - nP - L)), & x \in [nP + L, (n+1)P], \end{cases}$$

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Kirchhoff boundary conditions yield

$$\begin{cases} c_n = a_n \cos(\omega L) + b_n \sin(\omega L), \\ 2d_n = -a_n \sin(\omega L) + b_n \cos(\omega L), \end{cases}$$

and

$$\begin{cases} a_{n+1} = c_n \cos(\omega\pi) + d_n \sin(\omega\pi), \\ b_{n+1} = -2c_n \sin(\omega\pi) + 2d_n \cos(\omega\pi). \end{cases}$$

The monodromy matrix

$$M(\omega) := \begin{bmatrix} \cos(\omega\pi) & \sin(\omega\pi) \\ -2\sin(\omega\pi) & 2\cos(\omega\pi) \end{bmatrix} \begin{bmatrix} \cos(\omega L) & \sin(\omega L) \\ -\frac{1}{2}\sin(\omega L) & \frac{1}{2}\cos(\omega L) \end{bmatrix}$$

satisfies $\det(M) = 1$ and $\mathrm{tr}(M) = 2\cos(\omega\pi)\cos(\omega L) - \frac{5}{2}\sin(\omega\pi)\sin(\omega L)$.

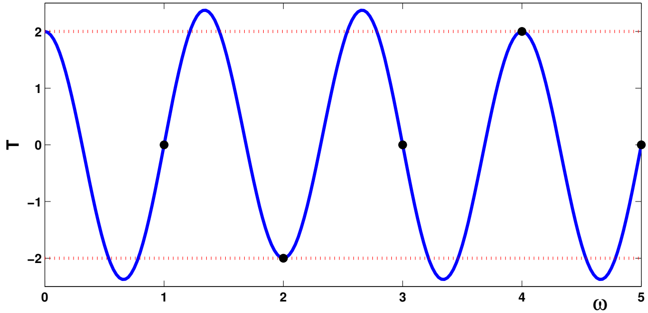
The symmetric part of the spectrum

Trace of the monodromy matrix:

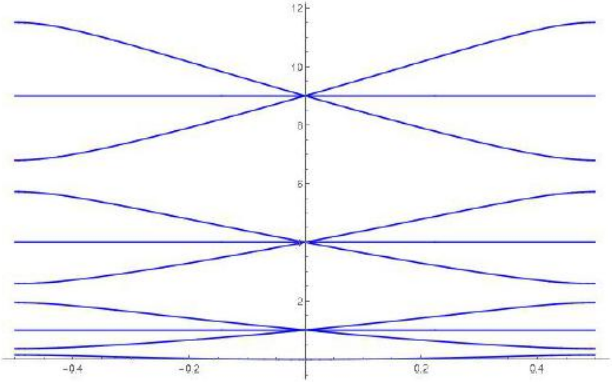
$$T(\omega) = 2\cos(\omega\pi)\cos(\omega L) - \frac{5}{2}\sin(\omega\pi)\sin(\omega L) \in [-2,2].$$

Note that $T(m) = 2(-1)^m \cos(mL) \in [-2,2]$ for every $m \in \mathbb{N}$.

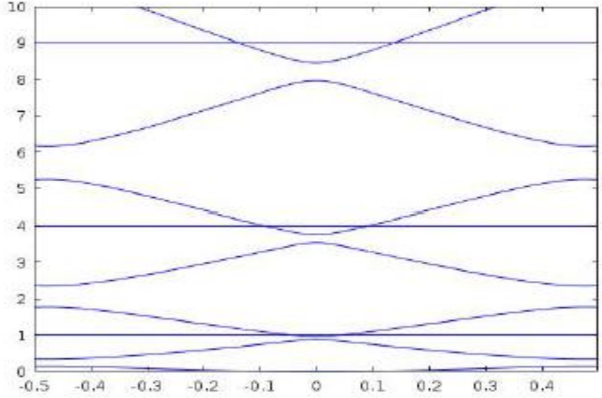
The spectrum $\sigma(-\partial_x^2)$ in $L^2(\Gamma)$ consists of eigenvalues $\{m^2\}_{m\in\mathbb{N}}$ of infinite multiplicity and a countable set of spectral bands $\{\sigma_k\}_{k\in\mathbb{N}}$. Moreover,
$m^2 \in \cup_{k\in\mathbb{N}}\sigma_k$ for every $m \in \mathbb{N}$.



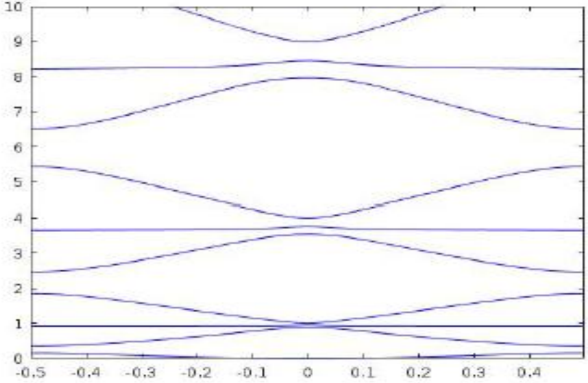
Numerical approximation of spectral bands: $L = \pi$



Numerical approximation of spectral bands: $L > \pi$



Numerical approximation of spectral bands: semi-rings of different lengths



The NLS equation on the periodic graph

Define piecewise functions for solutions of the NLS equation on the periodic graph Γ:

$$u_0(x) = \cup_{n\in\mathbb{Z}} \begin{cases} u_{n,0}(x), & x \in I_{n,0} = [2\pi n, 2\pi n + \pi], \\ 0, & \text{elsewhere}, \end{cases}$$

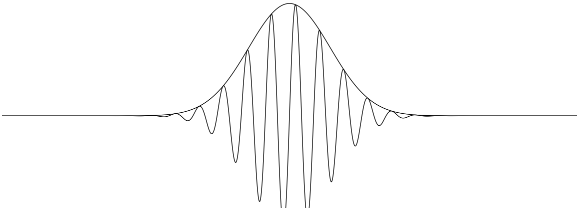
and

$$u_\pm(x) = \cup_{n\in\mathbb{Z}} \begin{cases} u_{n,\pm}(x), & x \in I_{n,\pm} = [2\pi n + \pi, 2\pi(n+1)], \\ 0, & \text{elsewhere}. \end{cases}$$

The NLS equation on the periodic graph Γ can be written as the evolutionary problem for $U = (u_0,\ u_+, u_-)$:

$$i\partial_t U + \partial_x^2 U + |U|^2 U = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\},$$

subject to the Kirchhoff boundary conditions at the vertex points.



Homogeneous NLS equation

The asymptotic solution in the form

$$U(t,x) = \varepsilon A(T,X) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} + \text{higher-order terms},$$

with $T = \varepsilon^2 t$ and $X = \varepsilon(x - c_g t)$ satisfies the homogeneous NLS equation

$$i\partial_T A + \frac{1}{2}\partial_\ell^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A + \nu |A|^2 A = 0, \quad \nu = \frac{\|f^{(m_0)}(\ell_0,\cdot)\|^4_{L^4_{\text{per}}}}{\|f^{(m_0)}(\ell_0,\cdot)\|^2_{L^2_{\text{per}}}}.$$

Theorem (Gilg–Schneider-P, 2016)

Fix $m_0 \in \mathbb{N}$, $\ell_0 \in \mathbb{T}_1$, and assume $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ for every $m \neq m_0$. Then, for every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A \in C(\mathbb{R}, H^3(\mathbb{R}))$ of the homogeneous NLS equation with

$$\sup_{T \in [0,T_0]} \|A(T,\cdot)\|_{H^3} \leq C_0$$

and for all $\varepsilon \in (0,\varepsilon_0)$, there are solutions $U \in C([0,T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ to the NLS equation on the periodic graph Γ satisfying the bound

$$\sup_{t \in [0,T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| U(t,x) - \varepsilon A(T,X) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C\varepsilon^{3/2}.$$

Function spaces

The operator $L = -\partial_x^2$ is considered in the space

$$\mathcal{L}^2 = \{U = (u_0, u_+, u_-) \in (L^2(\mathbb{R}))^3 : \quad \mathrm{supp}(u_{n,j}) = I_{n,j}, \quad n \in \mathbb{Z}, \; j \in \{0, +, -\}\}$$

with the domain of definition

$$\mathcal{H}^2 \quad := \quad \{U \in \mathcal{L}^2 : \; u_{n,j} \in H^2(I_{n,j}), \quad n \in \mathbb{Z}, \; j \in \{0, +, -\} \quad \text{Kirchhoff BCs}\}.$$

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- The space $\mathcal{H}^2$ is closed under pointwise multiplication.
- The skew symmetric operator $-iL$ defines a unitary semi-group $(e^{-iLt})_{t\in\mathbb{R}}$ in $\mathcal{L}^2$.
- There exists a positive constant C_L such that

  $$\|e^{-iLt}U\|_{\mathcal{H}^2} \leq C_L \|U\|_{\mathcal{H}^2}$$

  for every $U \in \mathcal{H}^2$ and every $t \in \mathbb{R}$.
- There exists a unique local solution $U \in C([-T_0, T_0], \mathcal{H}^2)$ to the NLS equation on the periodic graph Γ.

Bloch transform on the real line

For a function $f : \mathbb{R} \to \mathbb{C}$, Bloch transform is defined by

$$\widetilde{f}(\ell, x) = (\mathcal{T} f)(\ell, x) = \sum_{j \in \mathbb{Z}} f(x + 2\pi j) e^{-i\ell(x+2\pi j)}.$$

The inverse transform is

$$f(x) = (\mathcal{T}^{-1} \widetilde{f})(x) = \int_{-1/2}^{1/2} e^{i\ell x} \widetilde{f}(\ell, x) d\ell.$$

By construction, $\widetilde{f}(\ell, x)$ is extended from $(\ell, x) \in \mathbb{T}_1 \times \mathbb{T}_{2\pi}$ to $(\ell, x) \in \mathbb{R} \times \mathbb{R}$ according to the continuation conditions:

$$\widetilde{f}(\ell, x) = \widetilde{f}(\ell, x + 2\pi) \quad \text{and} \quad \widetilde{f}(\ell, x) = \widetilde{f}(\ell + 1, x) e^{ix}.$$

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- $\mathcal{T}$ is an isomorphism between $H^s(\mathbb{R})$ and $L^2(\mathbb{T}_1, H^s(\mathbb{T}_{2\pi}))$.
- Multiplication in x space corresponds to convolution in Bloch space.
- If $\chi : \mathbb{R} \to \mathbb{R}$ is 2π periodic, then

  $$\mathcal{T}(\chi u)(\ell, x) = \chi(x)(\mathcal{T} u)(\ell, x).$$

  In particular, if χ_j are periodic cut-off functions in I_j, $j \in \{0, +, -\}$, then $\mathcal{T}(u_j)(\ell, x) = \mathcal{T}(\chi_j u_j)(\ell, x) = \chi_j(x)(\mathcal{T} u_j)(\ell, x)$.

Function spaces for Bloch transforms

The operator $\tilde{L}(\ell) = -(\partial_x + i\ell)^2$ is self-adjoint in the space

$$L^2_\Gamma := \{ \widetilde{U} = (\widetilde{u}_0, \widetilde{u}_+, \widetilde{u}_-) \in (L^2(\mathbb{T}_{2\pi}))^3 : \quad \mathrm{supp}(\widetilde{u}_j) = I_{0,j}, \quad j \in \{0,+,-\}\}$$

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In Bloch space, we work with functions in $L^2(\mathbb{T}_1, L^2_\Gamma)$. Local well-posedness applies to smooth functions in $\widetilde{\mathcal{H}}^2 = L^2(\mathbb{T}_1, H^2_\Gamma)$.

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Key Lemma: *The Bloch transform $\mathcal{T}$ is an isomorphism between $\mathcal{H}^2$ and $\widetilde{\mathcal{H}}^2$.*

- Extend a piecewise H^2 function u_0 to $u_{0,ext} \in H^2(\mathbb{R})$.
- By Bloch transform on the real line, $\mathcal{T}(u_{0,ext}) \in L^2(\mathbb{T}_1, H^2(\mathbb{T}_{2\pi}))$.
- Compact support persists as $\widetilde{u}_0 = \mathcal{T}(u_0) = \mathcal{T}(\chi_0 u_{0,ext}) = \chi_0 \mathcal{T}(u_{0,ext})$.
- From the properties of $\mathcal{T}(u_{0,ext})$, we obtain $\widetilde{u}_0 \in L^2(\mathbb{T}_1, H^2(I_{0,0}))$.

Rest of the proof

- Bloch transform for the NLS equation on the periodic graph Γ.
- Decomposition of solutions in the Bloch space

$$\widetilde{U}(t,\ell,x) = \widetilde{V}(t,\ell) f^{(m_0)}(\ell,x) + \widetilde{U}^{\perp}(t,\ell,x)$$

- Approximation of the principal part of the solution

$$\widetilde{V}_{\text{app}}(t,\ell) = \widetilde{A}\left(\varepsilon^2 t, \frac{\ell-\ell_0}{\varepsilon}\right) e^{-i\omega^{(m_0)}(\ell_0)t} e^{-i\partial_\ell \omega^{(m_0)}(\ell_0)(\ell-\ell_0)t}.$$

  As $\varepsilon \to 0$, $\widetilde{A}$ satisfies the homogeneous NLS equation in the Fourier space.

- A near-identity transformation for $\widetilde{U}^{\perp}(t,\ell,x)$ with a suitable chosen approximation $\widetilde{U}^{\perp}_{\text{app}}(t,\ell,x)$.
- Estimates of residual terms in Bloch spaces.
- Estimates of the approximation between the Fourier space and Bloch space.
- Estimates of the error term in time evolution with Gronwall's inequality.

Bifurcations of nonlinear bound states

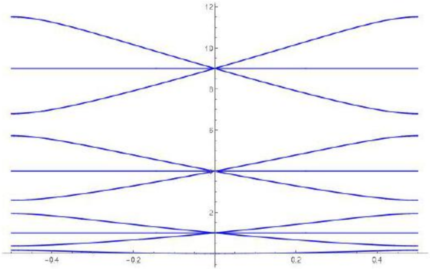
The stationary NLS equation on the periodic graph Γ:

$$-\partial_x^2 \phi - 2|\phi|^2 \phi = \Lambda \phi \qquad \Lambda \in \mathbb{R}, \quad \phi(x) : \Gamma \to \mathbb{R}.$$

The effective homogeneous NLS equation on the real line

$$-\frac{1}{2} \partial_\ell^2 \omega^{(m_0)}(\ell_0) \partial_X^2 A - \nu |A|^2 A = \Omega A, \quad A(X) : \mathbb{R} \to \mathbb{R}.$$

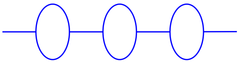
The stationary reduction is satisfied if $\partial_\ell \omega^{(m_0)}(\ell_0) = 0$.



Nonlinear bound states on the periodic graph

Stable bound states bifurcate from the bottom of the linear spectrum at $\Lambda = 0$:

$$-\partial_x^2 \phi - 2|\phi|^2 \phi = \Lambda \phi \qquad \Lambda \in \mathbb{R}, \quad \phi(x) : \Gamma \to \mathbb{R}.$$



Theorem

There are positive constants Λ_0 *and* C_0 *such that for every* $\Lambda \in (-\Lambda_0, 0)$, *there exist two bound states* $\phi \in \mathcal{D}(\Gamma)$ *(up to the discrete translational invariance) s.t. either*

$$\phi(x - L/2) = \phi(L/2 - x), \quad x \in \Gamma$$

or

$$\phi(x - L - \pi/2) = \phi(L + \pi/2 - x), \quad x \in \Gamma.$$

Moreover, it is true for both bound states that

(i) ϕ *is symmetric in upper and lower semicircles of* Γ,

(ii) $\phi(x) > 0$ *for every* $x \in \Gamma$,

(iii) $\phi(x) \to 0$ *as* $|x| \to \infty$ *exponentially fast.*

Numerical approximations of the bound states with $L = \pi$

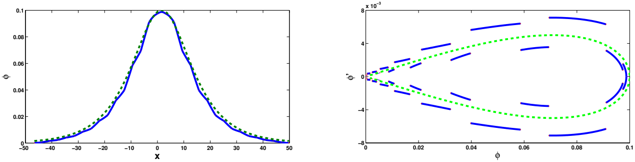


Figure : Profile of the numerically generated bound state on (x, ϕ) plane (left) and on (ϕ, ϕ') plane (right). The red dots show the break points on the periodic graph Γ. The green dashed line shows the NLS soliton on the infinite line.

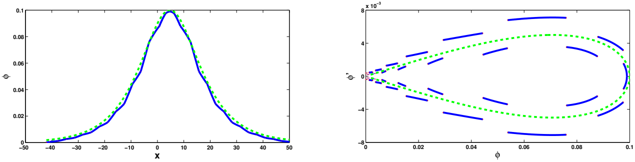


Figure : The same but for the other bound state.

Discrete homogenization method

We set $\Lambda = -\epsilon^2$ and consider the limit $\epsilon \to 0$.

For every $(a, b) \in \mathbb{R}^2$ and every $\epsilon \in \mathbb{R}$, there is a unique solution $\psi(x; a, b, \epsilon) \in C^\infty(\mathbb{R})$ of the initial-value problem:

$$\left\{ \begin{array}{ll} \partial_x^2 \psi - \epsilon^2 \psi + 2|\psi|^2 \psi = 0, & x \in \mathbb{R}, \\ \psi(0) = a, & \\ \partial_x \psi(0) = b, & \end{array} \right.$$

For each $\Gamma_{n,0}$ and $\Gamma_{n,\pm}$, the solution can be defined in the implicit form:

$$\phi_{n,0}(x) = \psi(x - nP; a_n, b_n, \epsilon), \quad \phi_{n,\pm}(x) = \psi(x - nP - L; c_n, d_n, \epsilon).$$

Kirchhoff boundary conditions produces a two-dimensional map:

$$\left\{ \begin{array}{l} a_{n+1} = \psi(\pi; c_n, d_n, \epsilon), \\ b_{n+1} = 2\partial_x \psi(\pi; c_n, d_n, \epsilon), \end{array} \right. \qquad \left\{ \begin{array}{l} c_n = \psi(L; a_n, b_n, \epsilon), \\ 2d_n = \partial_x \psi(L; a_n, b_n, \epsilon), \end{array} \right. \tag{2}$$

The nonlinear discrete map generalizes the linear transfer matrix method.

Approximate continuous solution

In the limit $\epsilon \to 0$, expand solution $\psi(x; \epsilon\alpha, \epsilon^2\beta, \epsilon)$ in the power series in ϵ. The two-dimensional map is now available in the perturbative form:

$$\begin{cases} \alpha_{n+1} = \alpha_n + \epsilon(L + \pi/2)\beta_n + \frac{1}{2}\epsilon^2(L^2 + \pi L + \pi^2)(1 - 2\alpha_n^2)\alpha_n + \mathcal{O}(\epsilon^3), \\ \beta_{n+1} = \beta_n + \epsilon(L + 2\pi)(1 - 2\alpha_n^2)\alpha_n + \frac{1}{4}\epsilon^2(2L^2 + 4L\pi + \pi^2)(1 - 6\alpha_n^2)\beta_n + \mathcal{O}(\epsilon^3). \end{cases}$$

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Approximate continuous solution:

$$\alpha_n = A(X+X_0), \quad \beta_n = B(X+X_0), \quad X = \epsilon n, \quad n \in \mathbb{Z},$$

where X_0 is arbitrary and A, B satisfy the continuous limit

$$\begin{cases} A'(X) = (L+\pi/2)B(X), \\ B'(X) = (L+2\pi)(1-2A^2)A(X), \end{cases}$$

with the continuous NLS solitons

$$A(X) = \mathrm{sech}(\nu X), \quad B(X) = -\mu \tanh(\nu X)\mathrm{sech}(\nu X), \quad X \in \mathbb{R},$$

Justification of the approximate continuous solution

Key Lemma: *For a given $f \in \ell^2(\mathbb{Z})$ satisfying the reversibility symmetry $f_n = f_{1-n}$ for every $n \in \mathbb{Z}$, consider solutions of the linearized difference equation*

$$-\frac{\alpha_{n+1} - 2\alpha_n + \alpha_{n-1}}{\epsilon^2} + \nu^2(1 - 6A^2(\epsilon n))\alpha_n = f_n, \quad n \in \mathbb{Z}.$$

For sufficiently small $\epsilon > 0$, there exists a unique solution $\alpha \in \ell^2(\mathbb{Z})$ satisfying the reversibility symmetry $\alpha_n = \alpha_{1-n}$ for every $n \in \mathbb{Z}$. Moreover there is a positive ϵ-independent constant C such that

$$\epsilon^{-1} \|\sigma_+\alpha - \alpha\|_{\ell^2} \leq C\|f\|_{\ell^2}, \quad \|\alpha\|_{\ell^2} \leq C\|f\|_{\ell^2},$$

where σ_+ is the shift operator defined by $(\sigma_+\alpha)_n := \alpha_{n+1}$, $n \in \mathbb{Z}$.

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where σ_+ *is the shift operator defined by* $(\sigma_+\alpha)_n := \alpha_{n+1}$*,* $n \in \mathbb{Z}$*.*

- Translational parameter X_0 can be chosen to satisfy the reversibility symmetry.
- Two reversibility symmetries give two nonlinear bound states.
- The symmetry $\phi_+ = \phi_-$ holds by construction.
- Positivity and exponential decay are not obtained from this method.

Positivity and exponential decay

The perturbative two-dimensional map:

$$\begin{cases} \alpha_{n+1} = \alpha_n + \epsilon(L+\pi/2)\beta_n + \frac{1}{2}\epsilon^2(L^2+\pi L+\pi^2)(1-2\alpha_n^2)\alpha_n + \mathcal{O}(\epsilon^3), \\ \beta_{n+1} = \beta_n + \epsilon(L+2\pi)(1-2\alpha_n^2)\alpha_n + \frac{1}{4}\epsilon^2(2L^2+4L\pi+\pi^2)(1-6\alpha_n^2)\beta_n + \mathcal{O}(\epsilon^3). \end{cases}$$

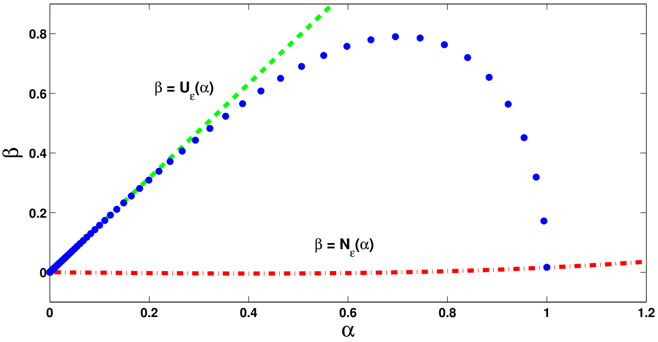

Figure : The plane (α, β), where the blue dots denote a sequence $\{\alpha_n, \beta_n\}_{n\in\mathbb{Z}}$, the green dashed line shows the unstable curve $\beta = \mathcal{U}_\epsilon(\alpha)$, and the red dash-dotted line shows the symmetry curve $\beta = \mathcal{N}_\epsilon(\alpha)$.

Conclusion

- We have defined the NLS evolution equations on graphs and considered the role of Kirchhoff boundary conditions in the energy conservation.
- We have justified the homogeneous NLS equation on the periodic graphs.
- We approximated nonlinear bound states near the lowest spectral band by using NLS solitons.
- We used discrete maps and dynamical system methods to study linear spectrum of the periodic graph Γ and the nonlinear bound states on Γ.
- Scattering and nonlinear dynamics on the periodic graph are still to be analyzed in some future.

Thank you!