Krein signature in $\mathcal{PT}$-symmetric systems

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- Stability in infinite-dimensional Hamiltonian systems
  - M. Grillakis, J. Shatah, W. Strauss (1990);
  - T. Kapitula, P.G. Kevrekidis, B. Sanstede (2004);
  - S. Cuccagna, D.P., V. Vougalter (2005);
  - M. Haragus, T. Kapitula (2006);
  - M. Chugunova, D.P (2010);
  - A. Stefanov, T. Kapitula (2013);
  - many others.

Stability of critical points in Hamiltonian systems

Consider an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J\, H'(u), \quad u(t) \in X$$

where X is the phase space, $J : X \mapsto X$ is a skew-adjoint operator with a bounded inverse $J^{-1} = -J$, and $H : X \to \mathbb{R}$ is the Hamilton function.

- Assume existence of a critical point $u_0 \in X$ such that $H'(u_0) = 0$.
- Perform linearization $u(t) = u_0 + v e^{\lambda t}$, where λ is the spectral parameter and $v \in X$ satisfies the spectral problem

  $$JH''(u_0)v = \lambda v,$$

  where $H''(u_0) : X \to X$ is a self-adjoint Hessian operator.
- If there exists λ with $\mathrm{Re}(\lambda) > 0$ and $v \in X$, then u_0 is called *spectrally unstable*. Otherwise, u_0 is *spectrally stable*.

Main Question

Assume:

- The spectrum of $H''(u_0)$ is strictly positive except for finitely many negative and zero eigenvalues of finite multiplicity.
- The spectrum of $JH''(u_0)$ is purely imaginary except for finitely many unstable eigenvalues.
- Multiplicity of the zero eigenvalue of $JH''(u_0)$ is given by the number of parameters in u_0 (symmetries).

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and eigenvalues of $H''(u_0)$ in the spectral problem

$$JH''(u_0)v = \lambda v.$$

Orbital Stability Theorem for Hamiltonian Systems

Consider the spectral stability problem:

$$JH''(u_0)v = \lambda v, \quad v \in X,$$

under the same assumptions on J and $H''(u_0)$. Eigenvalues λ appear in pairs relative to the imaginary axis: λ and $-\bar{\lambda}$.

Stability Theorem (Grillakis–Shatah–Strauss, 1990)

Assume zero eigenvalue of $H''(u_0)$ of multiplicity N and related N symmetries/conserved quantities. If $H''(u_0)$ has no negative eigenvalues under N constraints, then $JH''(u_0)$ has no unstable eigenvalues and an orbit of u_0 is linearly and nonlinearly stable.

Negative Index Theorem for Hamiltonian Systems

Stability Theorem (Kapitula–Promislow, 2013)

Assume no symmetries/zero eigenvalues of $H''(u_0)$. Then,

$$N_{\mathrm{re}}(JH'') + N_{\mathrm{c}}(JH'') + N_{\mathrm{im}}^{-}(JH'') = N_{\mathrm{neg}}(H'') < \infty,$$

where

- N_{re} - number of real unstable eigenvalues;
- N_{c} - number of complex unstable eigenvalues;
- N_{im}^{-} - number of imaginary eigenvalues of negative Krein signature.

Definition (Krein signature)

Suppose that $\lambda \in i\mathbb{R}\backslash\{0\}$ is a simple isolated eigenvalue of JH'' with the eigenvector v. The quadratic form $\langle H''v, v\rangle_{L^2} = \lambda\langle J^{-1}v, v\rangle_{L^2}$ is nonzero and its sign is called the Krein signature of the eigenvalue λ.

Example: degree-2 Hamiltonian system

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(-\lambda_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for H has two positive and two negative eigenvalues.

Both oscillators are unstable:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = \lambda_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 - \lambda_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$N_{\rm re}(JH) = 2 = N_{\rm neg}(H)$$

Example: degree-2 Hamiltonian system

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \omega_2^2 x_2^2)$$

The quadratic form for H has two positive and two negative eigenvalues.

The two oscillators are nevertheless stable:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \omega_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$N_{\mathrm{im}}^-(JH) = 2 = N_{\mathrm{neg}}(H)$$

Example: degree-2 Hamiltonian system

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \omega^2 x_1 x_2$$

The quadratic form for H has two positive and two negative eigenvalues.

The two oscillators are unstable with a quadruplet of complex eigenvalues:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega^2 x_2, \\ \dot{y}_2 = -\omega^2 x_1, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega^2 x_2 = 0, \\ \ddot{x}_2 - \omega^2 x_1 = 0, \end{cases} \quad \Rightarrow \quad x_1^{(4)} + \omega^4 x_1 = 0.$$

Negative index count:

$$N_{\rm c}(JH) = 2 = N_{\rm neg}(H)$$

Properties of Krein quantity

Definition (Krein quantity)

Suppose that $\lambda \in i\mathbb{R}\backslash\{0\}$ is a simple isolated eigenvalue of JH'' with the eigenvector v. The quadratic form

$$K(\lambda) := \langle H''v, v\rangle_{L^2} = \lambda\langle J^{-1}v, v\rangle_{L^2}$$

is called the Krein quantity of the eigenvalue λ.

Lemma (Krein quantity properties)

Suppose that $\lambda \in \mathbb{C}\backslash\{0\}$ is a simple isolated eigenvalue of JH''. Then:

1. $K(\lambda_0) \in \mathbb{R}$.
2. $K(\lambda_0) \neq 0$ if $\lambda_0 \in i\mathbb{R}$.
3. $K(\lambda_0) = 0$ if $\lambda_0 \in \mathbb{C}\backslash\{i\mathbb{R}\}$.

Necessary condition for instability bifurcation

Consider a perturbed spectral problem

$$J(H'' + \varepsilon W)v = \lambda v, \quad v \in X, \quad (*)$$

where $\varepsilon \ll 1$ is a perturbation parameter and
W is a symmetric bounded operator in X.

Instability Theorem

Suppose $\lambda_1(\varepsilon), \lambda_2(\varepsilon)$ are eigenvalues of $(*)$ continuously depending on $\varepsilon \in \mathbb{R}$. If $\lambda_1, \lambda_2 \in i\mathbb{R}$ with $K(\lambda_1)K(\lambda_2) < 0$ for $\varepsilon < 0$ and λ_1, λ_2 coalesce at $\varepsilon = 0$, then, under a certain non-degeneracy condition, $\lambda_1(\varepsilon), \lambda_2(\varepsilon)$ are complex for $\varepsilon > 0$.

Linear $\mathcal{PT}$-symmetric systems

Definition ($\mathcal{P}$ and $\mathcal{T}$ operators)

Parity transformation $\mathcal{P}$ and time reversal transformation $\mathcal{T}$:

$$\mathcal{P}u(x,t) := u(-x,t), \quad \mathcal{T}u(x,t) := \overline{u(x,-t)}.$$

Definition

A linear operator $L : X \to X$ is $\mathcal{PT}$-symmetric if it commutes with $\mathcal{PT}$:

$$[L, \mathcal{PT}] = L\mathcal{PT} - \mathcal{PT}L = 0.$$

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A $\mathcal{PT}$-symmetric operator L may have only real eigenvalues.

- $\mathcal{PT}$ symmetry in quantum mechanics (C.M. Bender, 1998)
- $\mathcal{PT}$-symmetry in nonlinear optics (D.N. Christodoulides *et al.* 2008)

Examples of $\mathcal{PT}$-symmetric operators

Consider a Schrödinger operator on $X = L^2(\mathbb{R})$:

$$L = -\partial_x^2 + V(x), \quad \text{where} \quad \bar{V}(-x) = V(x).$$

- a harmonic oscillator with a linear damping term

  $$V(x) = x^2 + 2i\gamma x = (x + i\gamma)^2 + \gamma^2$$

  The spectrum of L is purely discrete and real

  $$\sigma(L) = \left\{\gamma^2 + 1 + 2m, \quad m \in \mathbb{N}_0\right\}.$$

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  The spectrum of L is purely discrete and real

  $$\sigma(L) = \left\{\gamma^2 + 1 + 2m, \quad m \in \mathbb{N}_0\right\}.$$

- an unharmonic oscillator

  $$V(x) = x^2(-ix)^\gamma.$$

  The spectrum of L is purely discrete and real for $\gamma > 0$ (C.M. Bender, S.Boettcher 1998).

Properties of linear $\mathcal{PT}$-symmetric systems

Consider the evolution system

$$i\frac{du}{dt} = Lu, \quad u(\cdot, t) \in X,$$

where $L\mathcal{PT} - \mathcal{PT}L = 0$.

If $u(t)$ is a solution of the evolution equation, then

$$v(t) = \mathcal{PT}u(t) = \mathcal{P}\bar{u}(-t)$$

is also a solution of the same system.

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Lemma

If μ is an eigenvalue and U is an eigenfunction, then $\bar{\mu}$ is also an eigenvalue with the eigenfunction $\mathcal{PT}U$:

$$u(t) = Ue^{-i\mu t} \quad \Rightarrow \quad v(t) = \mathcal{P}\bar{U}e^{-i\bar{\mu}t}.$$

Linear $\mathcal{PT}$-symmetric systems

Consider a spectral problem for the $\mathcal{PT}$-symmetric linear operator L:

$$Lv = \mu v, \quad v \in X,$$

where $L\mathcal{PT} - \mathcal{PT}L = 0$.

Theorem (S.Nixon, J.Yang, 2016)

The spectral problem can be written in the Hamiltonian form

$$JHv = \lambda v,$$

where $J = i\mathcal{P}$, $H = \mathcal{P}L$, and $\lambda = i\mu$.

Proof: $(i\mathcal{P})(\mathcal{P}L)v = i\mu v$,
$H^* = L^*\mathcal{P} = \mathcal{P}L = H$,
$J^* = -\mathcal{P}i = -J$.

Krein quantity in linear $\mathcal{PT}$-symmetric systems

The spectral problem with the $\mathcal{PT}$-symmetric L:

$$Lv = \mu v \quad \Leftrightarrow \quad (i\mathcal{P})(\mathcal{P}L)v = i\mu v.$$

Definition (Krein quantity)

Suppose that $\mu \in \mathbb{R}\backslash\{0\}$ is a simple isolated eigenvalue of L with the eigenvector v. The Krein quantity of the eigenvalue μ is

$$K(\mu) := \langle \mathcal{P}Lv, v\rangle = \mu\langle \mathcal{P}v, v\rangle$$

Lemma (Krein quantity properties)

Suppose that $\mu \in \mathbb{C}\backslash\{0\}$ is a simple isolated eigenvalue of L. Then:

1. $K(\mu_0) \in \mathbb{R}$.
2. $K(\mu_0) \neq 0$ if $\mu_0 \in \mathbb{R}$.
3. $K(\mu_0) = 0$ if $\mu_0 \in \mathbb{C}\backslash\{\mathbb{R}\}$.

Stability of the linear $\mathcal{PT}$-symmetric systems

The spectral problem for the $\mathcal{PT}$-symmetric linear operator L:

$$Lv = \mu v \quad \Leftrightarrow \quad (i\mathcal{P})(\mathcal{P}L)v = i\mu v$$

with

$$L = -\partial_x^2 + x^2 + 2i\gamma x, \quad L = -\partial_x^2 + x^2(-ix)^\gamma.$$

For $\gamma = 0$: L is positive with $\mu > 0$, but $\mathcal{P}L$ has ∞-many eigenvalues of positive Krein signature and ∞-many eigenvalues of negative Krein signature:

$$K(\mu) = \langle \mathcal{P}Lv, v\rangle = \mu\langle \mathcal{P}v, v\rangle.$$

- Orbital Stability Theorem - NO
- Negative Index Theorem - NO
- Instability Bifurcation Theorem - YES

Infinitely many eigenvalues may become unstable.

Example of a discrete Schrödinger equation

Consider the spatially extended PT-symmetric potential,

$$-(u_{n+1}+u_{n-1})+\left(n^2+2i\gamma n\right)u_n=\mu u_n,\quad n\in\mathbb{Z}.$$

By using the discrete Fourier transform, the spectral problem is transformed to the differential equation

$$\frac{d^2\hat{u}}{dk^2}+2\gamma\frac{d\hat{u}}{dk}+[\mu+2\cos(k)]\,\hat{u}(k)=0,$$

subject to the 2π-periodicity of $\hat{u}(k)$.

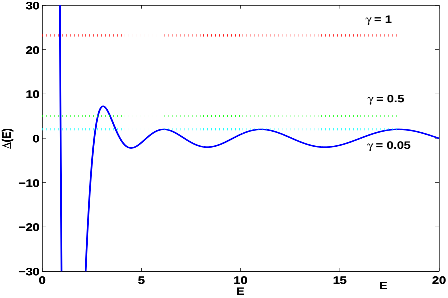
(D.P, P.Kevrekidis, D.Frantzeskakis, 2013)

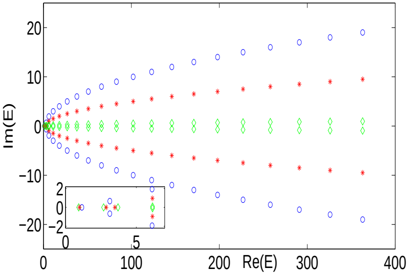
Example of a discrete Schrödinger equation

If $\hat{v}(k) = \hat{u}(k)e^{\gamma k}$, then $\hat{v}(k)$ satisfies the Mathieu equation:

$$\frac{d^2\hat{v}}{dk^2} + \left[\mu - \gamma^2 + 2\cos(k)\right]\hat{v} = 0,$$

subject to the condition $\hat{v}(k+2\pi) = e^{2\pi\gamma}\hat{v}(k)$. Hence we look for the Floquet multiplier $\mu_* = e^{2\pi\gamma}$ of the monodromy matrix.





Nonlinear $\mathcal{PT}$-symmetric systems

Main Question: How to extend the Krein quantity and related results to nonlinear $\mathcal{PT}$-symmetric systems?

$$i\partial_t\psi = \left[-\partial_x^2 + V(x) + i\gamma W(x)\right]\psi \pm |\psi|^2\psi,$$

where $V, W : \mathbb{R} \to \mathbb{R}$: $V(x) = V(-x)$, $W(-x) = -W(x)$, e.g.

- Wadati potential: $V(x) = \text{sech}^2(x)$, $W(x) = \text{sech}(x)\tanh(x)$;
- Confining potential: $V(x) = x^2$, $W(x) = xe^{-x^2/2}$.

This scalar model is different from dimers (coupled NLS systems), where some progress has been done:

- N.Alexeeva-I.Barashenkov-Yu.Kivshar (2012,2017);
- M.Stanislavova–A. Stefanov (2017);
- A. Chernyavsky–D.P. (2016).

Spectral stability problem

Stationary state: $\psi(t,x) = \Phi(x)e^{-i\mu t}$, $\mu \in \mathbb{R}$, $\Phi : \mathbb{R} \to \mathbb{C}$.

$$\mu\Phi = \left[-\partial_x^2 + V(x) + i\gamma W(x)\right] \Phi \pm |\Phi|^2\Phi,$$

Φ satisfies the $\mathcal{PT}$ symmetry: $\Phi = \mathcal{PT}\Phi$ or $\Phi(x) = \bar{\Phi}(-x)$.

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Φ satisfies the $\mathcal{PT}$ symmetry: $\Phi = \mathcal{PT}\Phi$ or $\Phi(x) = \bar{\Phi}(-x)$.

Linearization near Φ:

$$\begin{aligned} \psi(t,x) &= e^{-i\mu t} \left[\Phi(x) + Y(x) e^{-\lambda t} \right], \\ \bar{\psi}(t,x) &= e^{i\mu t} \left[\bar{\Phi}(x) + Z(x) e^{-\lambda t} \right], \end{aligned}$$

where $\lambda \in \mathbb{C}$ is spectral parameter:

$$\begin{bmatrix} L_0 + i\gamma W - \mu + 2|\Phi|^2 & \Phi^2 \\ \overline{\Phi}^2 & L_0 - i\gamma W - \mu + 2|\Phi|^2 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = -i\lambda \sigma_3 \begin{bmatrix} Y \\ Z \end{bmatrix}, \quad (*)$$

where $L_0 = -\partial_x^2 + V$ and $\sigma_3 = \mathrm{diag}(1,-1)$.

Krein quantity

The spectral problem

$$\underbrace{\begin{bmatrix} L_0 + i\gamma W - \mu + 2|\Phi|^2 & \Phi^2 \\ \overline{\Phi}^2 & L_0 - i\gamma W - \mu + 2|\Phi|^2 \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} Y \\ Z \end{bmatrix} = -i\lambda\sigma_3 \begin{bmatrix} Y \\ Z \end{bmatrix}, \quad (*)$$

and the adjoint spectral problem

$$\underbrace{\begin{bmatrix} L_0 - i\gamma W - \mu + 2|\Phi|^2 & \Phi^2 \\ \overline{\Phi}^2 & L_0 + i\gamma W - \mu + 2|\Phi|^2 \end{bmatrix}}_{\mathcal{L}^*} \begin{bmatrix} Y_a \\ Z_a \end{bmatrix} = i\bar{\lambda}\sigma_3 \begin{bmatrix} Y_a \\ Z_a \end{bmatrix}, \quad (**)$$

where $L_0 = -\partial_x^2 + V$ and $\sigma_3 = \mathrm{diag}(1, -1)$.

Main problem: no relations between eigenvectors and adjoint eigenvectors.

Krein quantity

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where $L_0 = -\partial_x^2 + V$ and $\sigma_3 = \mathrm{diag}(1, -1)$.

Lemma

If $\lambda_0 \in i\mathbb{R}$ is simple, the eigenvectors are $\mathcal{PT}$-symmetric, e.g. $Y = \mathcal{PT}Y$ or $Y(x) = \bar{Y}(-x)$.

Krein quantity

The spectral problem

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where $L_0 = -\partial_x^2 + V$ and $\sigma_3 = \mathrm{diag}(1, -1)$.

Definition (Krein signature)

Let $\lambda_0 \in i\mathbb{R}\backslash\{0\}$ be a simple isolated eigenvalue of the problem $(*)$ with the eigenvector (Y, Z) and the adjoint eigenvector (Y_a, Z_a). The Krein signature of λ_0 is the sign of the Krein quantity

$$K(\lambda_0) := \left\langle \sigma_3 \left[\begin{array}{c} Y \\ Z \end{array} \right], \left[\begin{array}{c} Y_a \\ Z_a \end{array} \right] \right\rangle = \int_{\mathbb{R}} \left[Y(x)\overline{Y_a(x)} - Z(x)\overline{Z_a(x)} \right] dx.$$

Krein quantity

The spectral problem

$$\begin{bmatrix} L_0 + i\gamma W - \mu + 2|\Phi|^2 & \Phi^2 \\ \overline{\Phi}^2 & L_0 - i\gamma W - \mu + 2|\Phi|^2 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = -i\lambda\sigma_3 \begin{bmatrix} Y \\ Z \end{bmatrix}, \quad (*)$$

where $L_0 = -\partial_x^2 + V$ and $\sigma_3 = \mathrm{diag}(1, -1)$.

Lemma (Krein quantity properties)

Assume that there exists a simple isolated eigenvalue $\lambda_0 \in \mathbb{C}\backslash\{0\}$ of the spectral problem $(*)$.Then:

1. $K(\lambda_0) \in \mathbb{R}$.
2. $K(\lambda_0) \neq 0$ if $\lambda_0 \in i\mathbb{R}$.
3. $K(\lambda_0) = 0$ if $\lambda_0 \in \mathbb{C}\backslash\{i\mathbb{R}\}$.

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where $L_0 = -\partial_x^2 + V$ and $\sigma_3 = \mathrm{diag}(1, -1)$.

Theorem (Necessary conditions for instability bifurcation)

Suppose $\lambda_1(\varepsilon), \lambda_2(\varepsilon)$ are eigenvalues of $(*)$ continuously depending on $\varepsilon \in \mathbb{R}$. If $\lambda_1, \lambda_2 \in i\mathbb{R}$ with $K(\lambda_1)K(\lambda_2) < 0$ for $\varepsilon < 0$ and λ_1, λ_2 coalesce into a defective eigenvalue at $\varepsilon = 0$, then, under a certain non-degeneracy condition, $\lambda_1(\varepsilon), \lambda_2(\varepsilon)$ are complex for $\varepsilon > 0$.

Behind the proof.

Assume self-adjointness of $L_0 = -\partial_x^2 + V$ on $L^2(\mathbb{R})$ and $W \in L^\infty(\mathbb{R})$.

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Nonlinear problem:

$$\mu\Phi = \left[-\partial_x^2 + V(x) + i\gamma W(x)\right]\Phi \pm |\Phi|^2\Phi,$$

Assume existence of $\Phi \in H^2(\mathbb{R})$ with real-analytic dependence on (γ, μ).

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Spectral problem:

$$\underbrace{\begin{bmatrix} L_0 + i\gamma W - \mu + 2|\Phi|^2 & \Phi^2 \\ \overline{\Phi}^2 & L_0 - i\gamma W - \mu + 2|\Phi|^2 \end{bmatrix}}_{\mathcal{L}} \underbrace{\begin{bmatrix} Y \\ Z \end{bmatrix}}_{v} = -i\lambda\sigma_3 \begin{bmatrix} Y \\ Z \end{bmatrix}, \quad (*)$$

Assume existence of a double defective eigenvalue λ_0 for (γ_0, μ_0) with eigenvector v_0 and generalized eigenvector v_0':

$$\mathcal{L}_0 v_0 = -i\lambda_0\sigma_3 v_0, \quad \mathcal{L}_0 v_1 = -i\lambda_0\sigma_3 v_1 - i\sigma_3 v_0.$$

Behind the proof.

Fix μ. The operator family $\mathcal{L} : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is real-analytic in γ at γ_0 with

$$\mathcal{L} = \mathcal{L}_0 + (\gamma - \gamma_0)\mathcal{L}_1 + \mathcal{O}((\gamma - \gamma_0)^2).$$

with $\mathcal{L}_1 : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$.

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with $\mathcal{L}_1 : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$.

Since λ_0 is a defective eigenvalue, use the Puiseux expansions:

$$\begin{aligned} \lambda &= \lambda_0 + (\gamma - \gamma_0)^{1/2}\lambda_1 + (\gamma - \gamma_0)\lambda_2 + \mathcal{O}((\gamma - \gamma_0)^{3/2}), \\ v &= v_0 + (\gamma - \gamma_0)^{1/2}v_1 + (\gamma - \gamma_0)v_2 + \mathcal{O}((\gamma - \gamma_0)^{3/2}). \end{aligned}$$

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with $\mathcal{L}_1 : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$.

Since λ_0 is a defective eigenvalue, use the Puiseux expansions:

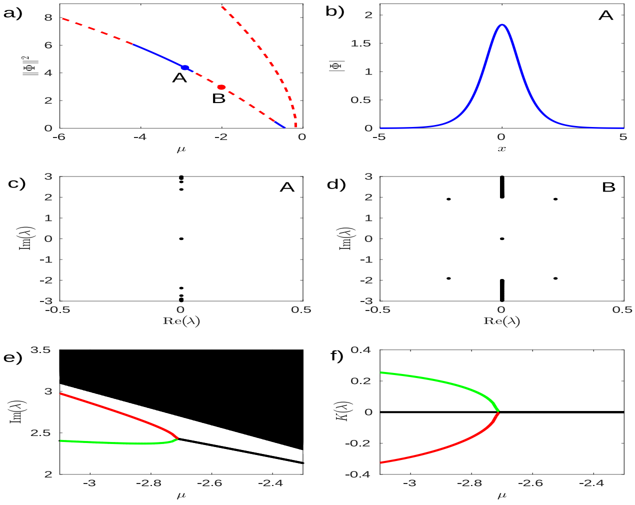
$$\begin{aligned} \lambda &= \lambda_0 + (\gamma - \gamma_0)^{1/2}\lambda_1 + (\gamma - \gamma_0)\lambda_2 + \mathcal{O}((\gamma - \gamma_0)^{3/2}), \\ v &= v_0 + (\gamma - \gamma_0)^{1/2} v_1 + (\gamma - \gamma_0) v_2 + \mathcal{O}((\gamma - \gamma_0)^{3/2}). \end{aligned}$$

Fredholm theory gives

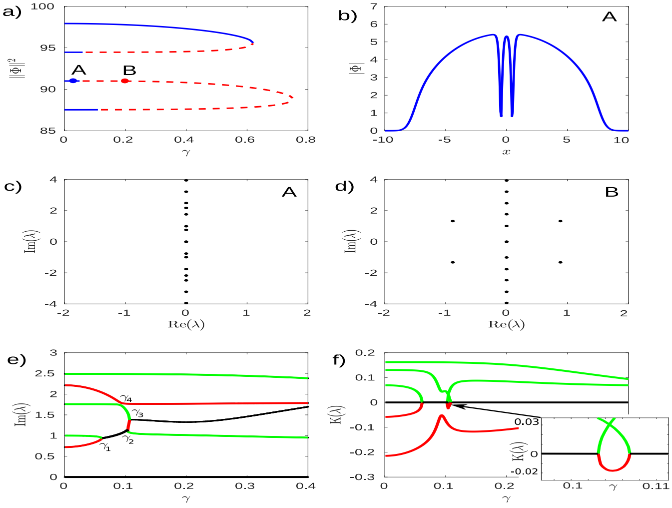
$$(-i\lambda_1)^2 = \frac{\langle \mathcal{L}_1 v_0, v_{0a} \rangle}{\sigma_3 \langle v_1, v_{0a} \rangle}.$$

The inner products are real and the splitting takes place if $\langle \mathcal{L}_1 v_0, v_{0a} \rangle \neq 0$. Justification is given by the Lyapunov-Schmidt reduction method.

Numerics: $V(x) = -2\,\mathrm{sech}^2\,x + i2.21\,\mathrm{sech}\,x\ \tanh x$



Numerical Results: $V(x) = x^2 + i\gamma x e^{-x^2}$



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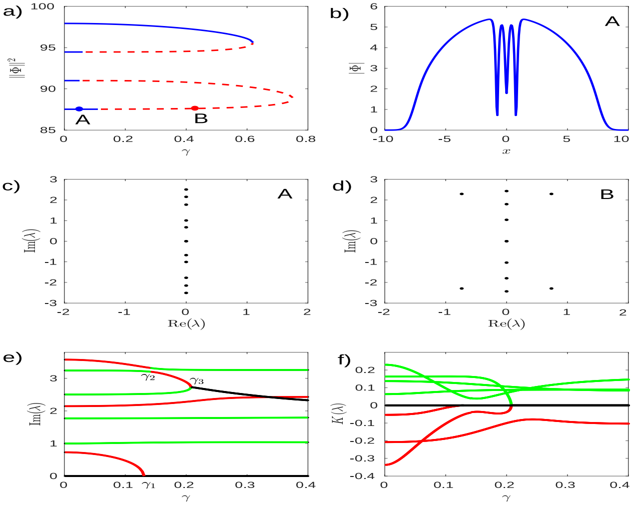

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Thank you!