Nonlinear Dirac equations and stability of solitary waves in one spatial dimension

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The problem

The nonlinear Dirac equations in one spatial dimension,

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u,v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u,v), \end{cases}$$

where $W(u,v) : \mathbb{C}^2 \to \mathbb{R}$ satisfies the following three conditions:

- symmetry $W(u,v) = W(v,u)$;
- gauge invariance $W(e^{i\theta}u, e^{i\theta}v) = W(u,v)$ for any $\theta \in \mathbb{R}$;
- polynomial in (u,v) and $(\bar{u},\bar{v})$.

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Compare with quantum relativistic physics in three spatial dimensions,

$$i\left(u_t + \sum_{j=1}^{3} \alpha_j u_{x_j}\right) - m\beta u + g(u\bar{u})\beta u = 0,$$

where $\bar{u} = \beta u^*$, $m \in \mathbb{R}$ is Dirac mass, $g(\cdot)$ is a nonlinear function, and

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix},$$

and $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices.

Examples in one dimension

- Coupled-mode equations for Bragg resonance (photonic crystals)

$$W = \alpha(|u|^2 + |v|^2)^2 + 2\alpha|u|^2|v|^2, \quad \alpha \in \mathbb{R}.$$

- Periodic modulations of Kerr nonlinearity (nonlinear optics)

$$W = \alpha(\bar{u}v + u\bar{v})(|u|^2 + |v|^2) + \beta(\bar{u}^2 v^2 + u^2 \bar{v}^2),$$

- Gross–Neveu model (general relativity)

$$W = \alpha(\bar{u}v + u\bar{v})^2,$$

- Massive Thirring model (integrable systems)

$$W = \alpha|u|^2|v|^2,$$

- Feshbach resonance in optical lattices (Bose–Einstein condensation)

$$W = \alpha(|u|^2 + |v|^2)|u|^2|v|^2.$$

Local existence and questions

Theorem

Assume $\mathbf{u}_0 \in H^s(\mathbb{R})$ *for any fixed* $s > \frac{1}{2}$. *There exists* $T > 0$ *such that the nonlinear Dirac equations*

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u, v), \end{cases}$$

admit a unique solution

$$\mathbf{u}(t) \in C([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R})) : \quad \mathbf{u}(0) = \mathbf{u}_0,$$

which depends continuously on the initial data.

- Global existence in $H^1(\mathbb{R})$ or even in $L^2(\mathbb{R})$:
  How does it depend on the nonlinearity W?
- Spectral stability of solitary waves:
  Can we control isolated eigenvalues inducing instabilities?
- Asymptotic stability of solitary waves:
  Is the linear dispersion sufficient for decay of perturbations?

Global existence in $H^1(\mathbb{R})$

Theorem (Delgado, 1978; Goodman, Weinstein and Holmes, 2001)

Assume that W is a polynomial in variables $|u|^2$ and $|v|^2$. A local solution is extended globally as $\mathbf{u}(t) \in C(\mathbb{R}_+, H^1(\mathbb{R}))$.

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- To obtain apriori energy estimates, cancellation of W is used in

$$\partial_t \left(|u|^{2p+2} + |v|^{2p+2}\right) + \partial_x \left(|u|^{2p+2} - |v|^{2p+2}\right) = i(p+1)(v\bar{u} - \bar{v}u)(|u|^{2p} - |v|^{2p}).$$

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- By Gronwall's inequality, we have

$$\|\mathbf{u}(t)\|_{L^{2p+2}} \leq e^{2|t|} \|\mathbf{u}(0)\|_{L^{2p+2}}, \quad t \in [0, T],$$

which holds for any $p \geq 0$ including $p \to \infty$.

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- This allows to control

$$\frac{d}{dt} \|\mathbf{u}_x(t)\|^2_{L^2} \leq C_W e^{4(N-1)|t|} \|\mathbf{u}_x(t)\|^2_{L^2},$$

where N is the degree of W in variables $|u|^2$ and $|v|^2$.

Discussion

- For the nonlinear Schrödinger equation,

$$iu_t + u_{xx} \pm |u|^{2N} u = 0,$$

global existence in $H^1(\mathbb{R})$ is known for $-|u|^{2N}u$ with any $N \geq 0$ and for $|u|^{2N}u$ with $0 \leq N < 2$. Blowup in a finite time is known for $N \geq 2$. For the nonlinear Dirac equation, the result does not depend on the power of nonlinearity.

- The energy conservation is crucial in the proof of global existence for the NLS equation and plays no role for the nonlinear Dirac equations, because the energy

$$H = \frac{i}{2}\int_{\mathbb{R}} (u_x\bar{u} - u\bar{u}_x - v_x\bar{v} + v\bar{v}_x)\, dx + \int_{\mathbb{R}} (v\bar{u} + u\bar{v} - W(u,v))\, dx$$

is not sign-definite near the zero equilibrium.

Global existence and scattering in Strichartz spaces

Strichartz spaces $L^p_t L^q_x$ and $L^q_x L^p_t$ are defined for $1 \leq p, q \leq \infty$ by

$$\|f\|_{L^p_t L^q_x} := \left(\int_0^T \|f(\cdot, t)\|^p_{L^q_x} dt \right)^{1/p}, \quad \|f\|_{L^q_x L^p_t} := \left(\int_{\mathbb{R}} \|f(x, \cdot)\|^q_{L^p_t} dx \right)^{1/q},$$

We say that a pair (q, r) is Strichartz admissible for the nonlinear Dirac equations in Strichartz space $L^q_t L^r_x$ if

$$q \geq 2, \quad r \geq 2 \quad \text{and} \quad \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}.$$

In particular, $(q, r) = (4, \infty)$ and $(q, r) = (\infty, 2)$ are end-point Strichartz pairs.

Lemma (Nakanishi, 1999; P, Stefanov, 2012)

Let (q, r) be a Strichartz admissible pair. There are constants $C > 0$ such that

$$\|e^{-it\mathcal{H}} \mathbf{f}\|_{L^4_t L^\infty_x} \leq C \|\mathbf{f}\|_{H^1_x},$$

$$\|e^{-it\mathcal{H}} \mathbf{f}\|_{L^\infty_t H^1_x} \leq C \|\mathbf{f}\|_{H^1_x},$$

$$\left\| \int_0^t e^{-i(t-\tau)\mathcal{H}} \mathbf{F}(\tau, \cdot) d\tau \right\|_{L^4_t L^\infty_x \cap L^\infty_t H^1_x} \leq C \|\mathbf{F}\|_{L^1_t H^1_x}.$$

Global existence and scattering in Strichartz spaces

Theorem

Assume W be a homogeneous polynomial in $\mathbf{u}$ of degree $2n+2$ for $n \geq 2$. Assume $\mathbf{u}(0) \in H^1(\mathbb{R})$ and $\|\mathbf{u}(0)\|_{H^1}$ be sufficiently small. There exists a global solution

$$\mathbf{u}(t) \in C(\mathbb{R}_+, H^1(\mathbb{R})) \cap L^4(\mathbb{R}_+, L^\infty(\mathbb{R})).$$

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$$\mathbf{u}(t) \in C(\mathbb{R}_+, H^1(\mathbb{R})) \cap L^4(\mathbb{R}_+, L^\infty(\mathbb{R})).$$

- Because $\|\mathbf{u}(t)\|_{L^\infty}$ is a continuous function of $t \in \mathbb{R}_+$ and $\|\mathbf{u}(t)\|_{L^\infty} \in L^4(\mathbb{R}_+)$, we have

$$\lim_{t\to\infty} \|\mathbf{u}(t)\|_{L^\infty} = 0.$$

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- Although cubic nonlinearity (quartic W with $n=1$) are excluded, analysis of Hayashi & Naumkin (2008,2009) relying on properties of $e^{-t\langle i\partial_x\rangle}$, where $\langle i\partial_x\rangle \equiv \sqrt{1-\partial_x^2}$, show scattering to zero with

$$\|\langle i\partial_x\rangle \mathbf{u}(t)\|_{L^\infty} \leq C\epsilon(1+t)^{-1/2}, \quad t \in \mathbb{R}_+,$$

if $\|\langle x\rangle\langle i\partial_x\rangle^4 \mathbf{u}(0)\|_{L^2} \leq \epsilon$ sufficiently small.

Proof of Theorem

- By Duhamel's principle, we have

$$\mathbf{u}(t) = e^{-it\mathcal{H}}\mathbf{u}(0) + \int_0^t e^{-i(t-s)\mathcal{H}}\nabla W(\mathbf{u}(s))ds,$$

  where $\mathcal{H}$ is the Dirac operator in one dimension.

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- By the preceding Lemma, we have

$$\|\mathbf{u}\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1} \leq C\|\mathbf{u}_0\|_{H^1} + C\|\nabla W(\mathbf{u})\|_{L_t^1 H_x^1},$$

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- By the assumption on nonlinearity,

$$\|\nabla W(\mathbf{u})\|_{L_t^1 H_x^1} \leq C\|(|\mathbf{u}| + |\mathbf{u}_x|)|\mathbf{u}|^{2n}\|_{L_t^1 L_x^2} \leq C\|\mathbf{u}\|_{L_t^\infty H_x^1}\|\mathbf{u}\|^{2n}_{L_t^{2n} L_x^\infty}.$$

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- For a Strichartz admissible pair (when $n \geq 2$),

$$\|\mathbf{u}\|_{L_t^{2n} L_x^\infty} \leq \|\mathbf{u}\|^{2/n}_{L_t^4 L_x^\infty}\|\mathbf{u}\|^{1-2/n}_{L_t^\infty L_x^\infty} \leq C\|\mathbf{u}\|_{L_t^4 L_x^\infty \cap L_t^\infty H_x^1}.$$

  The fixed point argument is closed for small $\mathbf{u}(0) \in H^1(\mathbb{R})$.

Existence of solitary waves

Time-periodic space-localized solutions

$$u(x,t) = U(x)e^{-i\omega t}, \quad v(x,t) = V(x)e^{-i\omega t}$$

satisfy a system of stationary Dirac equations

$$(\mathcal{H} - \omega I)\mathbf{U} + \nabla W(\mathbf{U}) = \mathbf{0}.$$

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- Translations in x and t can be added as free parameters.
- Constraint $\omega \in (-1,1)$ exists because spectrum of linear waves is located for $(-\infty,-1] \cup [1,\infty)$.
- If $|U|, |V| \to 0$ as $|x| \to \infty$, then $U(x) = \bar{V}(x)$ for all $x \in \mathbb{R}$.
- Analytical expressions are available for homogeneous polynomials W,

$$U(x) = \frac{\sqrt{1-\omega}}{\sqrt{1-\omega}\cosh(\sqrt{1-\omega^2}x) + i\sqrt{1+\omega}\sinh(\sqrt{1-\omega^2}x)}.$$

Given a time-periodic space-localized solution, the stability can be considered in three senses: (a) spectral, (b) orbital, and (c) asymptotic.

Spectral stability: We say that the gap soliton is spectrally unstable if the spectral problem for the linearized operator in $L^2(\mathbb{R})$ has at least one eigenvalue λ with $\mathrm{Re}\lambda > 0$. Otherwise, it is (weakly) spectrally stable.

Orbital stability: We say that the gap soliton $e^{-i\omega t}\mathbf{U}$ is orbitally stable if for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$, such that if $\|\mathbf{u}(0) - \mathbf{U}\|_{H^1} \leq \delta(\epsilon)$ then

$$\inf_{\theta \in \mathbb{R}} \|\mathbf{u}(t) - e^{-i\theta}\mathbf{U}\|_{H^1} \leq \epsilon,$$

for all $t > 0$.

Asymptotic stability: We say that the gap soliton is asymptotically stable if it is orbitally stable and for any $\mathbf{u}(0)$ near $\mathbf{U}$, there is $\mathbf{U}_\infty$ near $\mathbf{U}$ such that

$$\lim_{t \to \infty} \inf_{\theta \in \mathbb{R}} \|\mathbf{u}(t) - e^{-i\theta}\mathbf{U}_\infty\|_{L^\infty} = 0.$$

Spectral stability

Stability depends on W and ω. Linearization with the decomposition

$$\left\{ \begin{array}{l} u(x,t) = e^{-i\omega t}\left[U(x) + U_1(x)e^{\lambda t}\right], \\ v(x,t) = e^{-i\omega t}\left[V(x) + V_1(x)e^{\lambda t}\right], \end{array} \right.$$

yields the linear eigenvalue problem

$$\left\{ \begin{array}{rl} i\lambda \mathbf{U}_1 = & (\mathcal{H} - \omega I)\mathbf{U}_1 + V_{11}\mathbf{U}_1 + V_{12}\mathbf{U}_2, \\ -i\lambda \mathbf{U}_2 = & (\bar{\mathcal{H}} - \omega I)\mathbf{U}_2 + \bar{V}_{12}\mathbf{U}_1 + \bar{V}_{11}\mathbf{U}_2, \end{array} \right.$$

where $\mathbf{U}_{1,2} = [U_{1,2}, V_{1,2}]^T \in \mathbb{C}^2$, or compactly

$$\lambda \sigma \mathbf{Z} = H_\omega \mathbf{Z},$$

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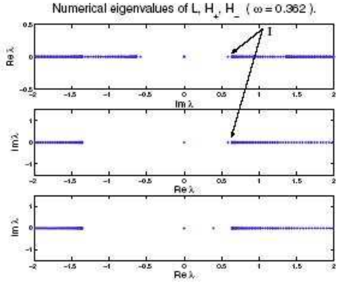
Lemma (Chugunova, P, 2006)

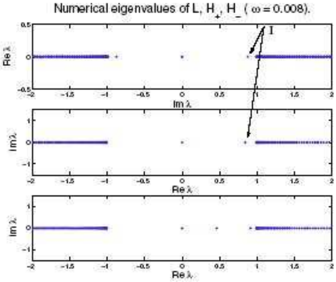
There exists an orthogonal similarity transformation S in $\mathbb{C}^4$ such that

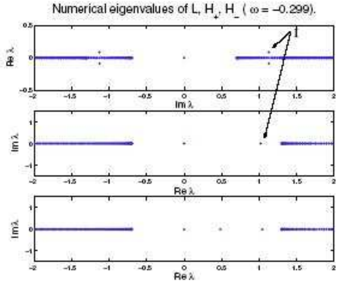
$$S^{-1} H_\omega S = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \qquad S^{-1} \sigma H_\omega S = \sigma \begin{pmatrix} 0 & H_- \\ H_+ & 0 \end{pmatrix}$$

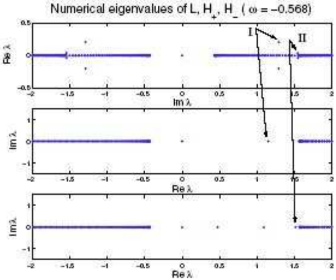
where $H_\pm$ are Dirac operators.

An example for cubic nonlinearity









Numerical eigenvalues of L, H_+, H_- ($\omega = -0.888$).

Numerical eigenvalues of L, H_+, H_- ($\omega = -0.986$)

Discussion

- Berkolaiko & Comech (Math. Model. Nat. Phenom. **7**, 13–31 (2012)) showed more examples of instabilities and tried to capture unstable eigenvalues.
- Comech (math-ph/1107.1763) constructed examples of neutrally stable eigenvalues and discussed the instability bifurcations (via famous Vakhitov-Kolokolov/Grillakis-Shatah-Strauss criterion)
- Comech (math-ph/1203.3859) studied bifurcations of eigenvalues from resonances of the nonlinear Schrödinger equations, which is an asymptotic reduction of the nonlinear Dirac equations
- Boussaid & Cuccagna (Comm. PDEs **37**, 1001–1056 (2012)) introduced a concept of Krein signature for eigenvalues of the Dirac operator with sign-indefinite energy and used it for asymptotic stability in three spatial dimensions.

Asymptotic stability of gap solitons

The nonlinear Dirac equations with a potential,

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u,v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u,v), \end{cases}$$

where $W = \beta(x)(|u|^2 + |v|^2) + \gamma(x)(\bar{u}v + u\bar{v}) + W_{\rm nl}(u,v)$.

Assumptions:

- $\beta, \gamma \in L^\infty(\mathbb{R})$ and there is $C > 0$ and $\kappa > 0$ such that

$$|\beta(x)| + |\gamma(x)| \leq C e^{-\kappa |x|}, \quad x \in \mathbb{R}.$$

- $\sigma(\mathcal{H}) \backslash \sigma_c(\mathcal{H}) = \{\omega_0\}$, where $\omega_0 \in (-1,1)$ is a simple eigenvalue of $\mathcal{H}$ with the L^2-normalized eigenfunction $\mathbf{u}_0 \in H^1(\mathbb{R})$.
- No resonances occur at the end points ± 1 of $\sigma_c(\mathcal{H})$ in the sense that no solutions of $\mathcal{H}\mathbf{u} = \pm \mathbf{u}$ exist in $L^\infty(\mathbb{R})$.
- The nonlinearity is homogeneous,

$$\nabla W_{\rm nl}(a\mathbf{U}) = a^{2n+1} \nabla W_{\rm nl}(\mathbf{U}), \quad a \in \mathbb{R}.$$

Local bifurcation

Lemma

Let Assumptions be satisfied and

$$\langle \mathbf{u}_0, \nabla W_{\rm nl}(\mathbf{u}_0)\rangle_{L^2} > 0.$$

For sufficiently small $\epsilon > 0$, there is a family of solutions $\mathbf{U} \in H^1(\mathbb{R})$ of the nonlinear Dirac equations for any $\omega \in (\omega_0, \omega_0 + \epsilon)$ such that the map $(\omega_0, \omega_0 + \epsilon) \ni \omega \mapsto \mathbf{U} \in H^1(\mathbb{R})$ is defined implicitly by small parameter $a \in \mathbb{R}$ and by the asymptotic expansion,

$$\begin{aligned} \|\mathbf{U} - a\mathbf{u}_0\|_{H^1} &= \mathcal{O}(a^{2n+1}), \\ |\omega - \omega_0 - a^{2n}\langle \mathbf{u}_0, \mathbf{N}(\mathbf{u}_0)\rangle_{L^2}| &= \mathcal{O}(a^{4n}), \end{aligned}$$

as $a \to 0$.

The proof holds by the Lyapunov–Schmidt decomposition,

$$\mathbf{U} = a\mathbf{u}_0 + \mathbf{V}, \quad a \in \mathbb{R}, \quad \langle \mathbf{u}_0, \mathbf{V}\rangle_{L^2} = 0.$$

Decomposition of the solution

Let us consider a local solution near the gap solitons,

$$\begin{cases} u(x,t) = e^{-i\theta(t)}\left[U(x;\omega(t)) + U_1(x,t)\right], \\ v(x,t) = e^{-i\theta(t)}\left[V(x;\omega(t)) + V_1(x,t)\right]. \end{cases}$$

If $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$, then $\mathbf{U}_1 = [U_1, V_1]^T$ satisfies the time evolution,

$$i\frac{d\mathbf{U}_1}{dt} = (\mathcal{H} - \omega I)\mathbf{U}_1 - i\dot{\omega}\partial_\omega \mathbf{U} - (\dot{\theta} - \omega)(\mathbf{U} + \mathbf{U}_1) + \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}),$$

where $\mathbf{N}(\mathbf{U}) = \nabla W_{\rm nl}(\mathbf{U})$.

Question: How to ensure that the decomposition is unique and to define evolutions of (ω, θ)?

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$$\left\{ \begin{array}{l} u(x,t) = e^{-i\theta(t)} \left[U(x;\omega(t)) + U_1(x,t) \right], \\ v(x,t) = e^{-i\theta(t)} \left[V(x;\omega(t)) + V_1(x,t) \right]. \end{array} \right.$$

If $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$, then $\mathbf{U}_1 = [U_1, V_1]^T$ satisfies the time evolution,

$$i \frac{d\mathbf{U}_1}{dt} = (\mathcal{H} - \omega I)\mathbf{U}_1 - i\dot{\omega}\partial_\omega \mathbf{U} - (\dot{\theta} - \omega)(\mathbf{U} + \mathbf{U}_1) + \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}),$$

where $\mathbf{N}(\mathbf{U}) = \nabla W_{\text{nl}}(\mathbf{U})$.

Question: How to ensure that the decomposition is unique and to define evolutions of (ω, θ)?

Answer: $\mathbf{U}_1$ is required to satisfy the symplectic orthogonality conditions to the two-dimensional generalized null space of the linearized operator.

Generalized null space of the linearized operator

The linearized operator

$$\lambda \sigma \mathbf{Z} = H_\omega \mathbf{Z},$$

where $\mathbf{Z} = [\mathbf{U}_1, \bar{\mathbf{U}}_1]^T \in \mathbb{C}^4$.

The kernel of the linearized operator:

$$\mathrm{Ker}(H_\omega) = \mathrm{span}\{\mathbf{F}\}, \quad \mathbf{F} = i \begin{bmatrix} \mathbf{U} \\ -\bar{\mathbf{U}} \end{bmatrix}, \quad H_\omega \mathbf{F} = \mathbf{0}.$$

The generalized kernel of the linearized operator

$$N_g(L_\omega) = \mathrm{span}\{\mathbf{F}, \mathbf{G}\}, \quad \mathbf{G} = -\partial_\omega \begin{bmatrix} \mathbf{U} \\ \bar{\mathbf{U}} \end{bmatrix}, \quad H_\omega \mathbf{G} = \sigma \mathbf{F}.$$

The symplectic orthogonality conditions are

$$\langle \sigma \mathbf{F}, \mathbf{Z} \rangle_{L^2} = 0, \quad \langle \sigma \mathbf{G}, \mathbf{Z} \rangle_{L^2} = 0,$$

or equivalently

$$\mathrm{Re}\langle \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} = 0, \quad \mathrm{Im}\langle \partial_\omega \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} = 0.$$

Modulation equations

Thanks to symplectic orthogonality conditions, we obtain the modulation equations on $\omega(t)$ and $\theta(t)$:

$$\begin{cases} \dot{\omega}\mathrm{Re}\langle \partial_\omega \mathbf{U}, \mathbf{U} - \mathbf{U}_1 \rangle_{L^2} + (\dot{\theta} - \omega)\mathrm{Im}\langle \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} = \Omega_1, \\ \dot{\omega}\mathrm{Im}\langle \partial_\omega^2 \mathbf{U}, \mathbf{U}_1 \rangle_{L^2} + (\dot{\theta} - \omega)\mathrm{Re}\langle \partial_\omega \mathbf{U}, \mathbf{U} + \mathbf{U}_1 \rangle_{L^2} = \Omega_2, \end{cases}$$

where

$$\begin{aligned} \Omega_1 &= \mathrm{Im}\left[\langle \mathbf{U}, \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}) \rangle_{L^2} + \langle \bar{V}_{12}\bar{\mathbf{U}} - V_{11}\mathbf{U}, \mathbf{U}_1 \rangle_{L^2}\right], \\ \Omega_2 &= \mathrm{Re}\left[\langle \partial_\omega \mathbf{U}, \mathbf{N}(\mathbf{U} + \mathbf{U}_1) - \mathbf{N}(\mathbf{U}) \rangle_{L^2} - \langle V_{12}\partial_\omega \bar{\mathbf{U}} + V_{11}\partial_\omega \mathbf{U}, \mathbf{U}_1 \rangle_{L^2}\right] \end{aligned}$$

Modulation equations determine uniquely the time evolution of $\mathbf{U}_1$.

Theorem (P. & Stefanov, 2012)

Fix $\epsilon > 0$ *and* $\delta > 0$ *sufficiently small such that* $\theta(0) = 0$, $\omega(0) \in (\omega_0, \omega_0 + \epsilon)$, *and* $\mathbf{U}_1(0) \in B_\delta(H^1)$. *There exist* $\epsilon_0 > \epsilon$, $\theta_\infty \in \mathbb{R}$, $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$, $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$, *and*

$$\mathbf{U}_1(t) \in C(\mathbb{R}_+, H^1) \cap L^4(\mathbb{R}_+, L^\infty)$$

such that $(\omega, \theta)(t)$ *solve the modulation equations,* $\mathbf{U}_1(t)$ *solves the time evolution equation, and*

$$\lim_{t\to\infty}\left(\theta(t) - \int_0^t \omega(s)ds\right) = \theta_\infty, \quad \lim_{t\to\infty}\omega(t) = \omega_\infty, \quad \lim_{t\to\infty}\|\mathbf{U}_1(t)\|_{L^\infty} = 0.$$

The proof of this theorem brings together Strichartz estimates for nonlinear terms and Mizumachi estimates for quadratic, exponentially decaying terms.

Discussion : open problems

- Global existence in $H^1(\mathbb{R})$ or even in $L^2(\mathbb{R})$:
  Can the proof be extended for W that depend on $(\bar{u}v + u\bar{v})$?

- Spectral stability of solitary waves:
  Can we use the new ideas of Krein signature to control isolated eigenvalues inducing instabilities?

- Asymptotic stability of solitary waves:
  Can we prove asymptotic stability for the cubic nonlinearity $n = 1$?

- Integrable system: the massive Thirring model

$$\left\{ \begin{array}{l} i(u_t + u_x) + v = |v|^2 u, \\ i(v_t - v_x) + u = |u|^2 v. \end{array} \right.$$

  Candy (2011) proved local and global well-posedness in $L^2(\mathbb{R})$.
  Can we use a Bäcklund transformation to control nonlinear perturbations to solitary waves in L^2?
  See T. Mizumachi and D. Pelinovsky, "Bäcklund transformation and $L2$-stability of NLS solitons", IMRN **2012**, 2034–2067 (2012)