Shocks and Solitons in the Periodic Nonlinear Maxwell Equations

Dmitry Pelinovsky

Department of Mathematics, McMaster University

with

Gideon Simpson - University of Minnesota

Michael I. Weinstein - Columbia University

Shocks and Spatial Periodicity

Spatially Homogeneous System of Conservation Laws

$$\partial_t \mathbf{v} + \partial_x \mathbf{f}(\mathbf{v}) = 0$$

Smooth data generates a shock in finite time (Lax 64)

Periodically Varying System of Conservation Laws

$$\begin{aligned} \partial_t \mathbf{v} + \partial_x \mathbf{f}(x, \mathbf{v}) = 0 \\ \mathbf{f}(x + 2\pi, \mathbf{v}) \;=\; \mathbf{f}(x, \mathbf{v}) \end{aligned}$$

Can spatial periodicity stabilize shock formation?

Regularizing Shocks

- Diffusive regularization:

$$v_t + vv_x = \mu v_{xx}$$

- Dispersive regularization:

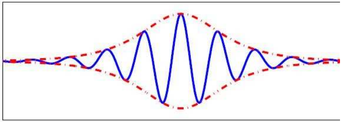
$$v_t + vv_x + \alpha v_{xxx} = 0$$

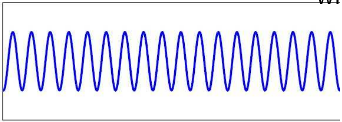
- Dispersion from Spatial Periodicity (Maxwell Model):

$$\partial_t^2 \left(n^2(z)E + \chi E^3\right) = \partial_z^2 E,$$
$$n^2(z + 2\pi) = n^2(z).$$

  - Does this model display wave breaking (shocks)?
  - Does this model admit stable localized states (solitons)?

Maxwell & Coupled Mode Equations





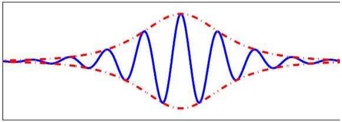
Periodic Nonlinear Maxwell Equation

$$\partial_t^2 \left(n^2(z) E + \chi E^3 \right) = \partial_z^2 E$$

where

$$n^2(z) = 1 + \epsilon \sum_{p \in \mathbb{Z}} N_p e^{ipz}, \quad \epsilon \ll 1.$$

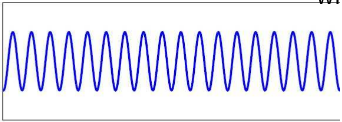
Maxwell & Coupled Mode Equations



Periodic Nonlinear Maxwell Equation

$$\partial_t^2 \left(n^2(z) E + \chi E^3 \right) = \partial_z^2 E$$

where



$$n^2(z) = 1 + \epsilon \sum_{p \in \mathbb{Z}} N_p e^{ipz}, \quad \epsilon \ll 1.$$

Two-wave approximation of small-amplitude resonant waves

$$E \approx \epsilon^{1/2} \left(\mathcal{E}^+(\epsilon z, \epsilon t) e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t) e^{-i(z+t)} \right)$$

yields the Nonlinear Coupled Mode Equations (NLCME) for $\mathcal{E}^{\pm}(Z, T)$ in slow variables $Z = \epsilon z$ and $T = \epsilon t$.

Properties of the NLCME

The Nonlinear Coupled Mode Equations (NLCME)

$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = iN_2 \mathcal{E}^- + i\Gamma \left(\left|\mathcal{E}^+\right|^2 + 2\left|\mathcal{E}^-\right|^2 \right) \mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\bar{N}_2 \mathcal{E}^+ + i\Gamma \left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2 \right) \mathcal{E}^-$$

- Dispersive, $\Omega^2 = K^2 + |N_2|^2$,
- Possess explicit solitary wave solutions (Aceves–Wabnitz 89),
- Globally well-posed in $H^1(\mathbb{R})$ (Goodman *et al.* 01), but

Properties of the NLCME

The Nonlinear Coupled Mode Equations (NLCME)

$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = iN_2 \mathcal{E}^- + i\Gamma\left(\left|\mathcal{E}^+\right|^2 + 2\left|\mathcal{E}^-\right|^2\right)\mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\bar{N}_2 \mathcal{E}^+ + i\Gamma\left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2\right)\mathcal{E}^-$$

- Dispersive, $\Omega^2 = K^2 + |N_2|^2$,
- Possess explicit solitary wave solutions (Aceves–Wabnitz 89),
- Globally well-posed in $H^1(\mathbb{R})$ (Goodman *et al.* 01), but
- Mathematically inconsistent, because the correction term $\tilde{\mathcal{E}}$,

$$\left(\partial_t^2 - \partial_z^2\right)\tilde{\mathcal{E}} = \left(\mathcal{E}^+\right)^3 e^{3i(z-t)} + \left(\mathcal{E}^-\right)^3 e^{-3i(z+t)} + \dots,$$

grow secularly in t.

NLCME Soliton Data and Numerics

Seed NLCME Soliton $(\mathcal{E}^+, \mathcal{E}^-)$ into Maxwell equations,

$$E(z,t) = \epsilon^{1/2}\left(\mathcal{E}^+(\epsilon z, \epsilon t)e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t)e^{-i(z+t)}\right).$$

- No periodic potential:

$$\partial_t^2\left(E + \chi E^3\right) = \partial_z^2 E$$

- Small cos-periodic potential:

$$\partial_t^2\left(E + \epsilon\cos(z)E + \chi E^3\right) = \partial_z^2 E$$

Side pulses are absent in the NLCME.

Revised Asymptotic Expansion

Hunter–Keller 83, Majda–Rosales 84, ...

Generalized Ansatz

$$E = \epsilon^{1/2}\left(E^{(0)}(z,t,Z,T) + \epsilon E^{(1)}(z,t,Z,T) + \dots\right).$$

Leading Order

$$E^{(0)} = E^{+}(z-t,Z,T) + E^{-}(z+t,Z,T)$$

Constraint on the Sublinear Growth of the Correction Term

$$\lim_{L\to\infty}\frac{1}{L}\int_0^L \left\| E^{(1)} \right\|(t)dt = 0.$$

Integro-Differential equations for $E^{\pm}(\phi, Z, T)$

$$\partial_T E^+ + \partial_Z E^+ = \partial_\phi \left\langle N(\phi+s)E^-(\phi+2s)\right\rangle_s + \Gamma\partial_\phi\left[\frac{1}{3}\left(E^+\right)^3 + E^+\left\langle\left(E^-\right)^2\right\rangle_s\right],$$

$$\partial_T E^- - \partial_Z E^- = -\partial_\phi \left\langle N(\phi-s)E^+(\phi-2s)\right\rangle_s - \Gamma\partial_\phi\left[\left\langle\left(E^+\right)^2\right\rangle_s E^- + \frac{1}{3}\left(E^-\right)^3\right]$$

where

$$\langle f\rangle_s = \lim_{L\to\infty}\frac{1}{L}\int_0^L f(s)ds.$$

Extended Nonlinear Coupled Mode Equations (xNLCMEs)

Periodically Varying Index of Refraction

$$N(z) = N(z + 2\pi) \quad \Rightarrow \quad N(z) = \sum N_p e^{ipz}, \quad N_0 = 0$$

Harmonic Decomposition

$$E^{\pm}(\phi, Z, T) = \sum E_p^{\pm}(Z, T) e^{ip\phi}.$$

Extended Nonlinear Coupled Mode Equations (xNLCMEs)

Periodically Varying Index of Refraction

$$N(z) = N(z + 2\pi) \quad \Rightarrow \quad N(z) = \sum N_p e^{ipz}, \quad N_0 = 0$$

Harmonic Decomposition

$$E^{\pm}(\phi, Z, T) = \sum E_p^{\pm}(Z, T) e^{ip\phi}.$$

xNLCMEs

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^- + \frac{\mathrm{i}p}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+ + \frac{\mathrm{i}p}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right]$$

Inclusion of third harmonic ($E^{\pm}_{\pm 3}$), resolves side pulses

Questions:

- Do the xNLCMEs admit localized stationary states (solitons)?
- If they do, are localized states robust in the time-dependent dynamics of the xNLCMEs?

Simplifications:

1. We reduce the system of xNLCMEs near band edges to a system of nonlinear Schrödinger equations.
2. We use the Gaussian trial functions and variational approximations.
3. We truncate the system of equations and perform parameter continuations.

Band Edge Approximation

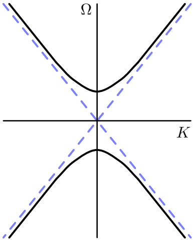
Localized stationary states

$$E_p^{\pm}(Z,T) = A_p^{\pm}(Z) e^{-ip\Omega T}, \quad A_p^{\pm}(Z) \sim e^{-|p||Z|\sqrt{|N_{2p}|^2-\Omega^2}}.$$

Assume $N_{2p} = 1$ for all p and $\Omega \in (-1,1)$.

Localized states near a band edge

$$A_p^{\pm}(Z) = \pm\mu U_p(\mu Z) + \mathrm{O}(\mu^2)$$
$$\Omega = \sqrt{1-\mu^2}, \quad 0 < \mu \ll 1.$$



This expansion allows us to derive coupled nonlinear Schrödinger equations.

Justification of the coupled NLS equations

Coupled Stationary Nonlinear Schrödinger Equation

$$U_p''(\zeta) - p^2 U_p + \frac{2}{3} p^2 \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0.$$

$$U(\theta, \zeta) = \sum U_p(\zeta) e^{ip\theta}$$

Theorem

Assume the existence of a localized state $U \in X^s$ of the NLS equations,

$$X^s \equiv \left\{ U(\zeta, \phi) \in H^s(\mathbb{R} \times \mathbb{T}) : \quad \bar{U}(\zeta, \phi) = U(\zeta, \phi), \right\}, \quad s > 1,$$

satisfying the symmetry $U_p(\zeta) = \bar{U}_p(-\zeta)$. There exists $\mu_0 > 0$ such that for any $|\mu| < \mu_0$, the xNLCMEs with $\Omega = \sqrt{1 - \mu^2}$ admit a unique localized state $A^{\pm} \in X^s$ satisfying the bound

$$\exists C > 0 : \quad \| A^{\pm} \mp \mu U(\mu \cdot, \cdot) \|_{X^s} \leq C \mu^2.$$

Existence of localized stationary states

Coupled NLS equations

$$U_p''(\zeta) - p^2 U_p + \frac{2}{3} p^2 \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0$$

Energy

$$H = \int_{\mathbb{R}} \sum \left(\frac{1}{p^2} |U_p'|^2 + |U_p|^2 \right) - \left(\sum |U_p|^2 \right)^2 - \frac{1}{3} \sum \bar{U}_p U_q U_r \bar{U}_{q+r-p} d\zeta.$$

Existence of localized stationary states

Coupled NLS equations

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3} p^2 \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0$$

Energy

$$H = \int_{\mathbb{R}} \sum \left(\frac{1}{p^2} |U_p'|^2 + |U_p|^2 \right) - \left(\sum |U_p|^2 \right)^2 - \frac{1}{3} \sum \bar{U}_p U_q U_r \bar{U}_{q+r-p} d\zeta.$$

Constrained variational problem

$$\text{minimize } H \ \text{ subject to fixed } N = \int_{\mathbb{R}} \sum |U_p|^2 d\zeta.$$

However, H is unbounded from below, even under the constraint.

Rayleigh–Ritz Approximation

Gaussian Ansatz

$$U_p(\zeta) = a_p e^{-b_p \zeta^2}, \quad p \in \mathbb{Z}_{\mathrm{odd}},$$

Reduced Energy

$$H_{\mathrm{G}} = \sum \frac{\sqrt{b_p} a_p^2}{p^2} + \frac{a_p^2}{\sqrt{b_p}} - \frac{a_p^2 a_q^2}{\sqrt{b_p + b_q}} - \frac{\sqrt{2} a_p a_q a_r a_{p-q-r}}{3\sqrt{b_p + b_q + b_r + b_{p-q-r}}}.$$

Euler–Lagrange Equations

$$\nabla_{\mathbf{a}} H_{\mathrm{G}}(\mathbf{a}, \mathbf{b}) = 0, \quad \nabla_{\mathbf{b}} H_{\mathrm{G}}(\mathbf{a}, \mathbf{b}) = 0.$$

Rayleigh–Ritz Approximation, Results

Truncated Solutions of Euler–Lagrange Equations:

No. of Modes	a_1	b_1	a_3	b_3	a_5	b_5
1	0.56060	0.33333	-	-	-	-
2	0.56321	0.33148	-0.13918	3.9413	-	-
3	0.56329	0.33189	-0.14585	3.6287	0.062822	8.5577

Questions:

- Does the solution converge to a localized state with finite N or H?
- Is the alternating sign between the modes important?
- Does the alternating sign persist with the number of modes?

Reduced Rayleigh–Ritz Approximation

Simplified Gaussian Ansatz

$$U_p(\zeta) = a_p e^{-b_p \zeta^2}, \quad p \in \mathbb{Z}_{\mathrm{odd}},$$

with

$$a_p = A(-1)^{(|p|-1)/2} \, |p|^{-\gamma}, \quad b_p = \frac{p^2}{3}$$

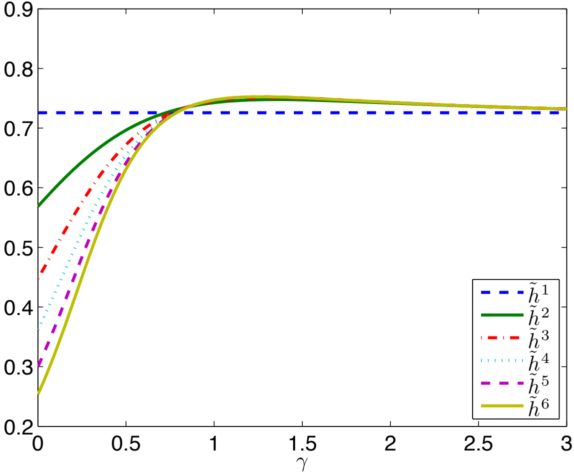
Two Parameter Energy

$$H_{\mathrm{G}} \equiv h_G(\gamma, A) = A^2 f(\gamma) - A^4 g(\gamma)$$

At a critical point, this expression simplifies to

$$h_G(\gamma, A(\gamma)) = \frac{f^2(\gamma)}{4g(\gamma)}$$

Reduced Rayleigh–Ritz Approximation, Results



$$\sum \|U_p\|_{L^2}^2 \sim \sum p^{-1-2\gamma}$$

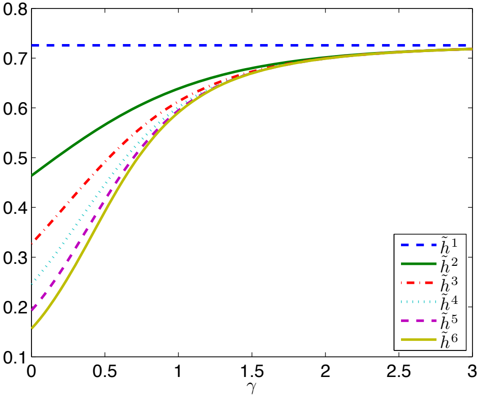
$$\gamma_\star \sim 1.26$$

$$U \in X^s, \quad 1 < s < 1.26$$

Ansatz without Alternating Signs

$$U_p(\zeta) = A\,|p|^{-\gamma}\, e^{-\frac{p^2}{3}\zeta^2}, \quad p \in \mathbb{Z}_{\mathrm{odd}},$$

No Extrema



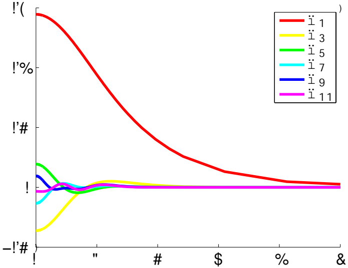
Direct Numerical Solution of Truncated NLS System

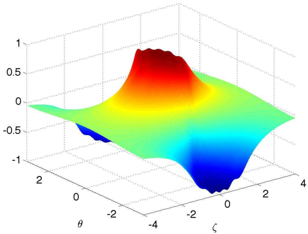
NLS System

$$U_p''(\zeta) - p^2 U_p + \frac{2}{3} p^2 \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0$$

$$U(\theta, \zeta) = \sum U_p(\zeta) e^{ip\theta}$$

Alternating Signs & # of Nodes, $|p| \le 12$





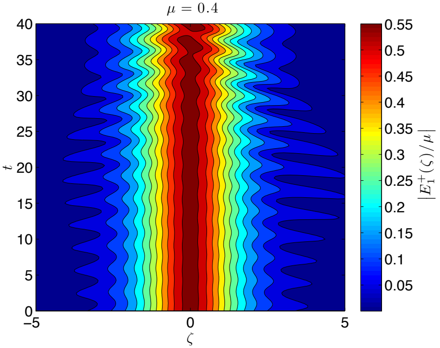
Persistence of Coupled NLS Solitons in xNLCMEs

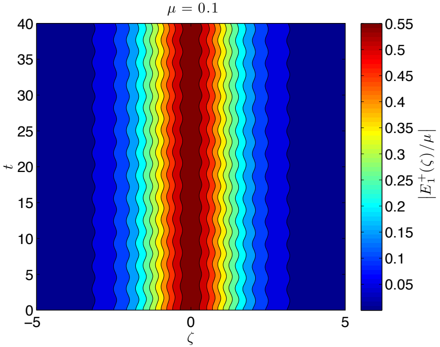
Resolves odd $|p| \leq 8$

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^- + \frac{\mathrm{i}p}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+ + \frac{\mathrm{i}p}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right].$$

$$E_p^{\pm}(Z,0) = \pm\mu U_p(\mu Z),\ p = 1$$





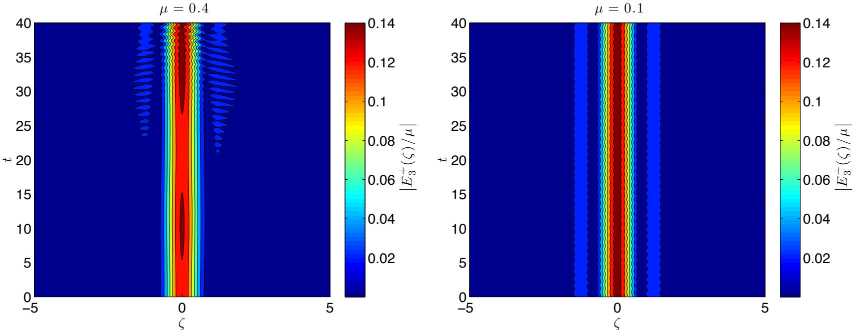
Persistence of Coupled NLS Solitons in xNLCMEs

Resolves odd $|p| \leq 8$

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^- + \frac{\mathrm{i}p}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+ + \frac{\mathrm{i}p}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right].$$

$$E_p^{\pm}(Z,0) = \pm\mu U_p(\mu Z),\ p = 3$$



Open question

Prove the existence of localized solutions in the nonlocal nonlinear elliptic problem:

$$(\partial_\zeta^2 + \partial_\theta^2) U = \frac{2}{3}\partial_\theta^2 \left[U^3 + 3 \left(\frac{1}{2\pi} \int |U|^2 \, d\theta \right) U \right].$$

where

$$U(\theta, \zeta) = \sum U_p(\zeta) e^{ip\theta}, \quad U : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$$

Summary:

Our results suggest that the localized states are robust for the nonlinear periodic Maxwell model. Existence of such states do not eliminate a possibility of shocks for large amplitudes.

References

- G. Simpson and M.I. Weinstein, “Coherent structures and carrier shocks in the nonlinear Maxwell equations”, *Multiscale Model Simul.* **9** (2011), 955–990.
- D.E. Pelinovsky, G. Simpson, and M.I. Weinstein, “Polychromatic solitary waves in a periodic and nonlinear Maxwell system”, *SIAM J. Appl. Dynam. Syst.* (2012), accepted.