Asymptotic stability of solitons in the discrete nonlinear Schrödinger equations

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References:

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P. Kevrekidis, D.P., A. Stefanov, SIAM J. Math. Anal. 41, 2010-2030 (2009)

The problem

The discrete nonlinear Schrödinger (DNLS) equation

$$i\dot{u}_n = (-\Delta + V_n)u_n + \gamma |u_n|^{2p} u_n = 0, \quad n \in \mathbb{Z},$$

where $\gamma = \pm 1$, $p \geq 1$, $V \in l^\infty(\mathbb{Z}, \mathbb{R})$, and

$$(\Delta u)_n := u_{n+1} - 2u_n + u_{n-1}.$$

Localized modes (time-periodic space-localized solutions) are of the form $u_n(t) = \phi_n e^{-i\omega t}$, where $\omega \in \mathbb{R}$ and $\{\phi_n\}_{n\in\mathbb{Z}}$ satisfies

$$\omega \phi_n = (-\Delta + V_n)\phi_n + \gamma |\phi_n|^2 \phi_n = 0, \quad n \in \mathbb{Z}.$$

Main Question: If a localized mode ϕ is orbitally stable, is it also asymptotically stable due to dispersive radiation?

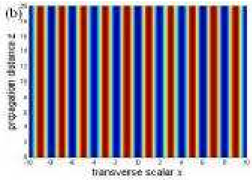
Physical contexts

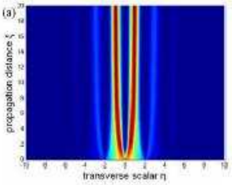
The DNLS equation arises in the modeling of density waves in Bose–Einstein condensates in the framework of the Gross–Pitaevskii equation

$$iu_t = -\nabla^2 u + V(x)u + \gamma|u|^2 u$$

with a bounded 2π-periodic potential $V(x) = V(x + 2\pi)$.

Another context of the DNLS equation is the coupled waveguide arrays in nonlinear optics and photorefractive crystals.



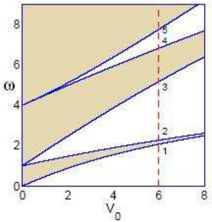


Existence of gap solitons

Localized modes of the Gross–Pitaevskii equation satisfy the stationary equation with a periodic potential

$$\omega\phi = -\nabla^2\phi + V(x)\phi + \gamma|\phi|^2\phi, \quad x \in \mathbb{R}^d.$$

Spectrum of $L = -\nabla^2 + V(x)$ for $V(x) = V_0 \sin^2(x)$, $d = 1$:

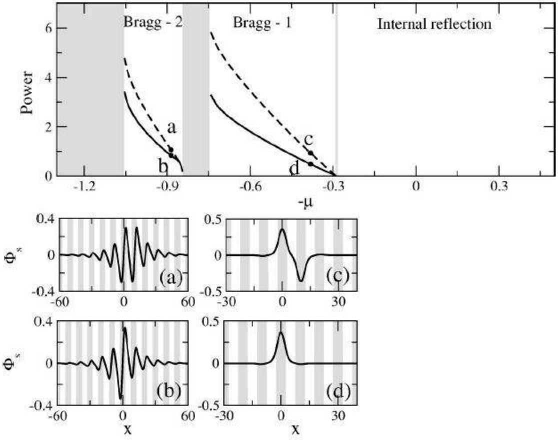


Theorem (Pankov, 2005)

Let V be a real-valued bounded periodic potential. Let ω be in a finite gap of the spectrum of $L = -\nabla^2 + V(x)$. There exists a non-trivial weak solution $U \in H^1(\mathbb{R}^d)$, which is continuous on $x \in \mathbb{R}^d$ and decays to 0 exponentially.

Numerical approximation of gap solitons

P., Sukhorukov, Kivshar (2004): $V(x) = \sin^2(x)$, $\gamma = +1$



Asymptotic reductions of the GP equation

The Gross–Pitaevskii equation can be reduced asymptotically with a multiple scale expansion method to one of the three models.

- Nonlinear Dirac equations for **small-amplitude** potentials

$$\left\{ \begin{array}{c} i(a_t + a_x) + b = \gamma(|a|^2 + 2|b|^2)a \\ i(b_t - b_x) + a = \gamma(2|a|^2 + |b|^2)b \end{array} \right.$$

  Goodman, Holmes, & Weinstein (2001); Schneider & Uecker (2001); P., Schneider (2007).

- Nonlinear Schrödinger equations near band edges

$$ia_t + a_{xx} + \gamma|a|^2 a = 0$$

  Busch (2006); Dohnal, P., Schneider (2009); Ilan & Weinstein (2010)

- Discrete nonlinear Schrödinger equations for **large-amplitude** potentials

$$i\dot{a}_n + \alpha(a_{n+1} + a_{n-1}) + \gamma|a_n|^2 a_n = 0.$$

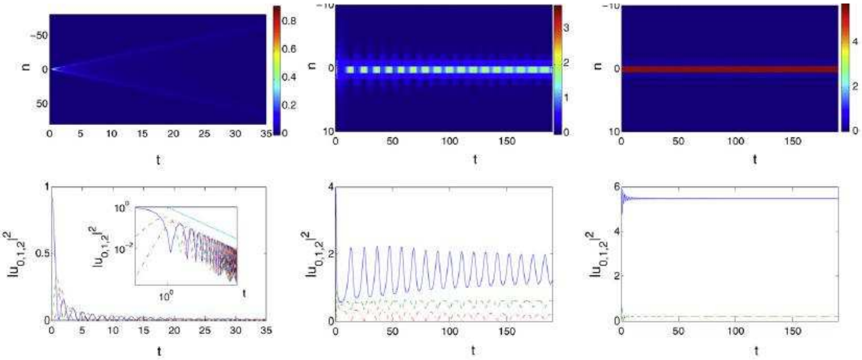
  P., Schneider, MacKay (2008); P., Schneider (2010)

Back to the problem of asymptotic stability

Kevrekidis et al. (2008)

$$i\dot{u}_n + u_{n+1} - 2u_n + u_{n-1} + |u_n|^2 u_n = 0$$

$u_n(0) = A\delta_{n,0}$ with $A = 1$ (left), $A = 2$ (middle), and $A = 2.5$ (right).



Given a time-periodic space-localized solution $\phi_n e^{-i\omega t}$ of the DNLS equation, the stability can be considered in the following three senses:
(a) spectral, (b) orbital, and (c) asymptotic.

Spectral stability: We say that the localized mode ϕ is spectrally unstable if the spectral problem for the linearized evolution in $l^2(\mathbb{Z})$ has at least one eigenvalue λ with $\mathrm{Re}\lambda > 0$. Otherwise, it is (weakly) spectrally stable.

Linearized evolution is found after the substitution

$$u_n(t) = e^{-i\omega t}\left(\phi_n + v_n e^{\lambda t} + i w_n e^{\lambda t}\right),$$

and neglection of the terms $\|\mathbf{v}\|_{l^2}^2$ and $\|\mathbf{w}\|_{l^2}^2$. Then, $(\mathbf{v}, \mathbf{w})$ satisfy the linear eigenvalue problem

$$L_+ \mathbf{v} = -\lambda \mathbf{w}, \quad L_- \mathbf{w} = \lambda \mathbf{v},$$

where $L_\pm$ are discrete Schrödinger operators with decaying potentials on $\mathbb{Z}$.

The 2-parameter orbit of the localized mode

$$e^{-i\omega t - i\theta}\boldsymbol{\phi},$$

where $\theta \in \mathbb{R}$ is an arbitrary parameter due to the phase rotation.

Orbital stability: The localized mode ϕ is said to be orbitally stable if for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$, such that if $\|\mathbf{u}(0) - \boldsymbol{\phi}\|_{l^2} \leq \delta(\epsilon)$ then

$$\inf_{\theta\in\mathbb{R}} \|\mathbf{u}(t) - e^{-i\theta}\boldsymbol{\phi}\|_{l^2} \leq \epsilon,$$

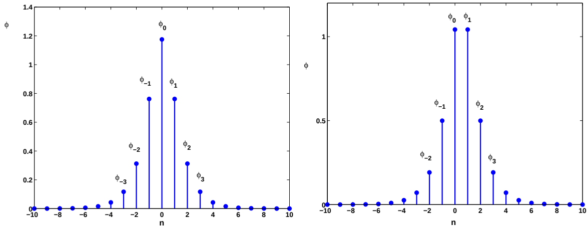
for all $t > 0$.

Asymptotic stability: The localized mode ϕ is said to be asymptotically stable if it is orbitally stable and for any $\mathbf{u}(0)$ near ϕ, there is ϕ_∞ near ϕ such that

$$\lim_{t\to\infty} \inf_{\theta\in\mathbb{R}} \|\mathbf{u}(t) - e^{-i\theta}\boldsymbol{\phi}_\infty\|_{l^2} = 0.$$

Spectral stability

Stability depends on ϕ. Consider, for example, two single-humped localized modes, existence of which can be proved for many DNLS equations:



For the cubic DNLS equation, the solution on the left is spectrally stable, whereas the solution on the right is spectrally unstable.

Note that both solitons are stable for the continuous NLS equation

$$iu_t + u_{xx} + |u|^2 u = 0,$$

where the localized mode is $\phi(x) = \sqrt{2\omega}\,\mathrm{sech}(\sqrt{\omega}(x-s))$, $s \in \mathbb{R}$.

More general formulation

Let us consider the DNLS equation in the form

$$i\dot{u}_n = (-\Delta + V_n)u_n + |u_n|^{2p}u_n, \quad n \in \mathbb{Z},$$

where $p \geq 1$ (an integer) and $V \in l^\infty(\mathbb{Z})$.

Assumptions on V:

- $V_n \to 0$ as $n \to \infty$ sufficiently fast, so that $V \in l^1_s(\mathbb{Z})$ with $s \geq 1$;
- V supports no resonances near the band edges of $\sigma(-\Delta) = [0,4]$;
- V supports exactly one negative eigenvalue $\omega_0 < 0$ of $H = -\Delta + V$ with an eigenvector $\boldsymbol{\psi}_0 \in l^2$ (normalized by $\|\boldsymbol{\psi}_0\|_{l^2} = 1$).

For instance, if $V_n = -\delta_{n,0}$, the assumption is satisfied with

$$(\boldsymbol{\psi}_0)_n = e^{-\kappa|n|}, \quad n \in \mathbb{Z},$$

where $\kappa = \operatorname{arcsinh}(2^{-1})$ and $\omega_0 = 2 - \sqrt{5} < 0$.

Lemma

Fix $\sigma \geq 0$. For any $\mathbf{u}_0 \in l^2_\sigma$, there exists a unique solution $\mathbf{u}(t) \in C^1(\mathbb{R}_+, l^2_\sigma)$ s.t. $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{u}(t)$ depends continuously on $\mathbf{u}_0$.

Local existence follows from the Picard iterations applied to

$$u_n(t) = u_n(0) - i \int_0^t \left[(-\Delta + V_n) u_n(t') + |u_n(t')|^{2p} u_n(t') \right] dt'$$

in space $C([0,T], l^2_\sigma)$. To show that $T = \infty$, we can use the balance equation

$$i \frac{d}{dt} |u_n|^2 = u_n(\bar{u}_{n+1} + \bar{u}_{n-1}) - \bar{u}_n(u_{n+1} + u_{n-1}),$$

so that

$$\|\mathbf{u}(t)\|^2_{l^2_\sigma} \leq \|\mathbf{u}(0)\|^2_{l^2_\sigma} + C \int_0^t \|\mathbf{u}(t')\|^2_{l^2_\sigma} dt'.$$

By Gronwall's inequality, $\|\mathbf{u}(t)\|^2_{l^2_\sigma}$ is bounded and continuous for any $t > 0$.

Local bifurcation of localized modes

Lemma

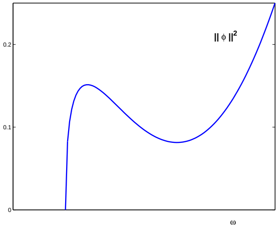
Let $\epsilon := \omega - \omega_0$. For any $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0 > 0$ is small, there exists a solution $\phi(\omega) \in C([\omega_0, \omega_0 + \epsilon_0), l^2)$ of

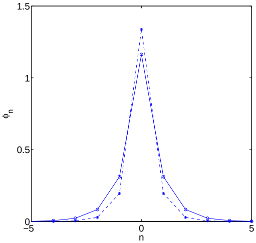
$$(-\Delta + V_n)\phi_n + \phi_n^{2p+1} = \omega\phi_n, \quad n \in \mathbb{Z},$$

satisfying

$$\left\| \phi - \frac{\epsilon^{\frac{1}{2p}} \boldsymbol{\psi}_0}{\|\boldsymbol{\psi}_0\|_{l^{2p+2}}^{1+\frac{1}{p}}} \right\|_{l^2} \leq C\epsilon^{1+\frac{1}{2p}}.$$

Moreover, $\{\phi_n\}_{n\in\mathbb{Z}}$ decays exponentially to zero as $|n| \to \infty$.





Orbital stability

Lemma

There exists an orbitally stable minimizer of energy

$$E(\mathbf{u}) = \sum_{n\in\mathbb{Z}} |u_{n+1} - u_n|^2 + V_n|u_n|^2 + \frac{1}{p+1}\gamma|u_n|^{2p+2}$$

under a fixed $N(\mathbf{u}) = \|\mathbf{u}\|_{l^2}^2 > 0$ *for any* $p \geq 1$.
If $p \geq 2$ *and* $V \equiv 0$*, then the minimizer only exists for* $N(\mathbf{u}) \geq N_0 > 0$.

Grillakis, Shatah, Strauss (1987,1990); Weinstein (1999);
Pankov (2006,2007).

If $\mathbf{u}(0) \approx \phi(\omega(0))$, then $\mathbf{u}(t)$ remains near $\phi(\omega(t))$ for all $t > 0$ and $|\omega(t) - \omega(0)|$ remains small. However, the question is if there exists ω_∞ so that $\mathbf{u}(t) \to \phi(\omega_\infty)$ and $\omega(t) \to \omega_\infty$ as $t \to \infty$.

Main result

Theorem

Let $p \geq 3$. Fix $\epsilon > 0$ and $\delta > 0$ be small and assume that $\omega(0) = \omega_0 + \epsilon$ and

$$\|\mathbf{u}(0) - \phi(\omega_0 + \epsilon)\|_{l^2} \leq \delta \epsilon^{\frac{1}{2p}}.$$

Under the three assumptions on V, there exist $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$, $(\omega, \theta) \in C^1(\mathbb{R}_+)$, and

$$\mathbf{y}(t) = \mathbf{u}(t) - e^{-i\theta(t)}\phi(\omega(t)) \in C^1(\mathbb{R}_+, l^2) \cap L^6(\mathbb{R}_+, l^\infty)$$

such that $\mathbf{u}(t)$ solves the DNLS equation and

$$\lim_{t\to\infty} \omega(t) = \omega_\infty, \quad \lim_{t\to\infty} \|\mathbf{u}(t) - e^{-i\theta(t)}\phi(\omega(t))\|_{l^\infty} = 0.$$

Remark: A similar result applies in the focusing case $\gamma = -1$ with the local bifurcation to $\omega < \omega_0$.

Earlier works on continuous NLS equations are by Soffer,Weinstein (1992), Pillet,Wayne (1997), Yao,Tsai (2002), Mizumachi (2008), Cuccagna (2008). Discrete setting: Cuccagna & Tarulli (2009), Kevrekidis, P., Stefanov (2009).

Decomposition of the solution

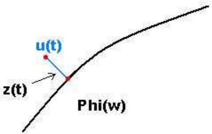
Let

$$\mathbf{u}(t) = e^{-i\theta(t)} \left(\boldsymbol{\phi}(\omega(t)) + \mathbf{z}(t) \right).$$

If $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$, then $\mathbf{z}(t) \in C^1(\mathbb{R}_+, l^2_\sigma)$ solves

$$i\dot{\mathbf{z}} = (H - \omega)\mathbf{z} - (\dot{\theta} - \omega)(\boldsymbol{\phi}(\omega) + \mathbf{z}) - i\dot{\omega}\partial_\omega \boldsymbol{\phi}(\omega) + \mathbf{N}(\boldsymbol{\phi}(\omega) + \mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)),$$

where $H = -\Delta + V$ and $[\mathbf{N}(\boldsymbol{\psi})]_n = |\psi_n|^{2p}\psi_n$.



Question: How to ensure that the decomposition is unique?

Double null space

Linearized time evolution for $\mathbf{z}(t) = \mathbf{v}(t) + i\mathbf{w}(t)$ is defined by the non-self-adjoint eigenvalue problem

$$L_+ \mathbf{v} = -\lambda \mathbf{w}, \quad L_- \mathbf{w} = \lambda \mathbf{v},$$

where

$$L_- = H - \omega + \phi_n^{2p}, \quad L_+ = H - \omega + (2p+1)\phi_n^{2p}.$$

If $\langle \boldsymbol{\phi}(\omega), \partial_\omega \boldsymbol{\phi}(\omega) \rangle_{l^2} \neq 0$, there exists a double zero eigenvalue with a one-dimensional kernel, isolated from the rest of the spectrum. The generalized kernel is spanned by vectors

$$[\mathbf{0}, \boldsymbol{\phi}(\omega)]^T, \quad [-\partial_\omega \boldsymbol{\phi}(\omega), \mathbf{0}]^T.$$

$(\mathbf{v}, \mathbf{w}) \in l^2$ is symplectically orthogonal to the double subspace of the generalized kernel under the conditions

$$\langle \mathbf{v}, \boldsymbol{\phi}(\omega) \rangle_{l^2} = 0, \quad \langle \mathbf{w}, \partial_\omega \boldsymbol{\phi}(\omega) \rangle_{l^2} = 0.$$

Symplectic orthogonality

Lemma

Fix $\epsilon \in (0, \epsilon_0)$. There exists $\delta > 0$ and $T > 0$ such that any $\mathbf{u} \in l^2$ satisfying

$$\|\mathbf{u} - \boldsymbol{\phi}(\omega_0 + \epsilon))\|_{l^2} \leq \delta \epsilon^{\frac{1}{2p}}$$

can be uniquely decomposed by

$$\mathbf{u} = e^{-i\theta} \left(\boldsymbol{\phi}(\omega) + \mathbf{z} \right)$$

and

$$\langle \mathrm{Re}\mathbf{z}, \boldsymbol{\phi}(\omega) \rangle_{l^2} = \langle \mathrm{Im}\mathbf{z}, \partial_\omega \boldsymbol{\phi}(\omega) \rangle_{l^2} = 0,$$

with $(\omega, \theta) \in \mathbb{R}^2$ and $\mathbf{z} \in l^2$. Moreover, there exists $C > 0$ such that

$$|\omega - \omega_0 - \epsilon| \leq C\delta\epsilon, \quad |\theta| \leq C\delta, \quad \|\mathbf{z}\|_{l^2} \leq C\delta\epsilon^{\frac{1}{2p}}.$$

The mapping $\mathbf{u} \mapsto (\omega, \theta, \mathbf{z})$ is a C^1 diffeomorphism.

Projections

The time-evolution of (ω, θ) satisfies the system

$$\mathbf{A}(\omega, \mathbf{z}) \begin{bmatrix} \dot{\omega} \\ \dot{\theta} - \omega \end{bmatrix} = \mathbf{f}(\omega, \mathbf{z}),$$

where

$$\mathbf{A}(\omega, \mathbf{z}) = \langle \boldsymbol{\phi}(\omega), \partial_\omega \boldsymbol{\phi}(\omega) \rangle_{l^2} I + \mathcal{O}(\|\mathbf{z}\|_{l^2}),$$

and

$$\|\mathbf{f}(\omega, \mathbf{z})\| \leq C \left(\langle \boldsymbol{\phi}^{2p-1}, \mathbf{z}^2 \rangle_{l^2} + \langle \boldsymbol{\phi}, \mathbf{z}^{2p+1} \rangle_{l^2} \right).$$

The time evolution of $\mathbf{z}(t)$ is governed by

$$i\dot{\mathbf{z}} = (H - \omega)\mathbf{z} - (\dot{\theta} - \omega)(\boldsymbol{\phi}(\omega) + \mathbf{z}) - i\dot{\omega}\partial_\omega \boldsymbol{\phi}(\omega) + \mathbf{N}(\boldsymbol{\phi}(\omega) + \mathbf{z}) - \mathbf{N}(\boldsymbol{\phi}(\omega)),$$

where

$$\|\mathbf{N}(\boldsymbol{\phi} + \mathbf{z}) - \mathbf{N}(\boldsymbol{\phi})\|_{l^\infty} \leq C \left(\|\boldsymbol{\phi}^{2p} \mathbf{z}\|_{l^\infty} + \|\mathbf{z}^{2p+1}\|_{l^\infty} \right).$$

Estimates on the discrete part

We need to show that $\dot\omega, \dot\theta - \omega \in L^1_t \cap L^\infty_t$ from the estimates like

$$\begin{aligned}\int_0^T |\dot\omega| dt &\leq C\epsilon^{2-\frac{1}{p}} \| < n >^{-2\sigma} |\mathbf{z}|^2 \|_{L^1_t l^\infty_n} \| < n >^{2\sigma} \phi \|_{L^\infty_t l^1_n} \\ &\leq C\epsilon^{2-\frac{1}{p}} \| < n >^{-\sigma} \mathbf{z} \|^2_{l^\infty_n L^2_t},\end{aligned}$$

for some fixed $\sigma > 0$.

If $\| < n >^{-\sigma} \mathbf{z} \|_{l^\infty_n L^2_t} \leq C\delta\epsilon^{\frac{1}{2p}}$, then

$$\|\omega - \omega_0 - \epsilon\|_{L^\infty} \leq C\delta^2\epsilon^2,$$

and there exists $\omega_\infty := \lim_{t\to\infty} \omega(t)$ such that $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$.

Moreover, we establish that $(\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ such that $\mathbf{z}(t) \in C^1(\mathbb{R}_+, l^2)$ by the global well-posedness. It remains to prove that

$$\| < n >^{-\sigma} \mathbf{z} \|_{l^\infty_n L^2_t} \leq C \|\mathbf{z}(0)\|_{l^2_n} \leq C\delta\epsilon^{\frac{1}{2p}}.$$

Dispersive decay estimates

Discrete pointwise estimates: There exists a constant $C > 0$ depending on V such that for all $t > 0$,

$$\begin{aligned} \left\| \langle n \rangle^{-\sigma} e^{-itH} P_c \mathbf{f} \right\|_{l^2_n} &\leq C(1+t)^{-3/2} \|\langle n \rangle^{\sigma} \mathbf{f}\|_{l^2_n}, \\ \left\| e^{-itH} P_c \mathbf{f} \right\|_{l^\infty_n} &\leq C(1+t)^{-1/3} \|\mathbf{f}\|_{l^1_n}. \end{aligned}$$

By the theory of Keel-Tao (1998), the pointwise estimates are transferred to the time averaged estimates.

Discrete Strichartz estimates: There exists a constant $C > 0$ such that

$$\begin{aligned} \left\| e^{-itH} P_c \mathbf{f} \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} &\leq C \|\mathbf{f}\|_{l^2_n}, \\ \left\| \int_0^t e^{-i(t-s)H} P_c \mathbf{g}(s) ds \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} &\leq C \|\mathbf{g}\|_{L^1_t l^2_n}, \end{aligned}$$

where

$$\|\mathbf{f}\|_{L^p_t l^q_n} = \left(\int_0^T \|\mathbf{f}(t)\|^p_{l^q_n} dt \right)^{1/p}, \quad \|\mathbf{f}\|_{l^q_n L^p_t} = \left(\sum_{n \in \mathbb{Z}} \|f_n\|^q_{L^p_t} \right)^{1/q}.$$

Estimates on the continuous part

Strichartz estimates provide a sufficient tool to treat the free solution and the nonlinear term in the integral equation for $\mathbf{z}(t)$,

$$\mathbf{z}(t) = e^{-itH} P_c \mathbf{z}(0) - i \int_0^t e^{-i(t-s)H} P_c (\mathbf{g}_1(s) + \mathbf{g}_2(s) + \mathbf{g}_3(s)) ds,$$

where

$$\mathbf{g}_1 = \mathbf{N}(\phi + \mathbf{y} e^{i\theta}) - \mathbf{N}(\phi), \quad \mathbf{g}_2 = -(\dot{\theta} - \omega)\phi, \quad \mathbf{g}_3 = -i\dot{\omega}\partial_\omega \phi(\omega).$$

We have

$$\| e^{-itH} P_c \mathbf{z}(0) \|_{L_t^6 l_n^\infty \cap L_t^\infty l_n^2} \leq C \|\mathbf{z}(0)\|_{l_n^2}$$

and

$$\left\| \int_0^t e^{-i(t-s)H} P_c |\mathbf{z}(s)|^{2p+1} ds \right\|_{L_t^6 l_n^\infty \cap L_t^\infty l_n^2} \leq C \| |\mathbf{z}|^{2p+1} \|_{L_t^1 l_n^2} \leq C \|\mathbf{z}\|^{2p+1}_{L_t^{2p+1} l_n^{2(2p+1)}}.$$

For any $p \geq 3$, the pair $(r, w) = ((2p+1), 2(2p+1))$ is the admissible Strichartz pair in the sense

$$\frac{6}{r} + \frac{2}{w} \leq 1,$$

so that

$$\|\mathbf{z}\|_{L_t^{2p+1} l_n^{2(2p+1)}} \leq \|\mathbf{z}\|_{L_t^6 l_n^\infty} + \|\mathbf{z}\|_{L_t^\infty l_n^2}.$$

Estimates on the continuous part

However, to deal with terms $|\phi|^{2p}|\mathbf{z}(s)|$ as well as with $\|\dot\omega\|_{L^1_t}$, we also need the estimates on $\| < n >^{-\sigma} \mathbf{z}\|_{l^\infty_n L^2_t}$. These estimates are obtained by Mizumachi (2008).

Discrete Mizumachi estimates: There exists a constant $C > 0$ such that

$$\begin{aligned}
\| < n >^{-3/2} e^{-itH} P_c \mathbf{f}\|_{l^\infty_n L^2_t} &\leq C\|\mathbf{f}\|_{l^2_n} \\
\left\| < n >^{-\sigma} \int_0^t e^{-i(t-s)H} P_c \mathbf{F}(s) ds \right\|_{l^\infty_n L^2_t} &\leq C\| < n >^{\sigma} \mathbf{F}\|_{l^1_n L^2_t} \\
\left\| < n >^{-\sigma} \int_0^t e^{-i(t-s)H} P_c \mathbf{F}(s) ds \right\|_{l^\infty_n L^2_t} &\leq C\|\mathbf{F}\|_{L^1_t l^2_n} \\
\left\| \int_0^t e^{-i(t-s)H} P_c \mathbf{F}(s) ds \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} &\leq C\| < n >^{3} \mathbf{F}\|_{L^2_t l^2_n}.
\end{aligned}$$

As a result, we obtain

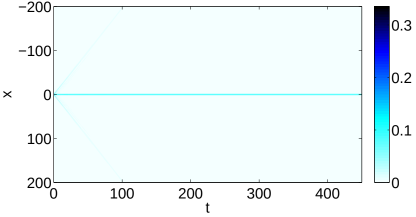
$$\begin{aligned}
\left\| \int_0^t e^{-i(t-s)H} P_c |\phi|^{2p} |\mathbf{z}(s)| ds \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} &\leq C\| < n >^{3} |\phi|^{2p} |\mathbf{z}| \|_{L^2_t l^2_n} \\
&\leq \| < n >^{3+\sigma} |\phi|^{2p} \|_{L^\infty_t l^2_n} \| < n >^{-\sigma} \mathbf{z}\|_{l^\infty_n L^2_t}.
\end{aligned}$$

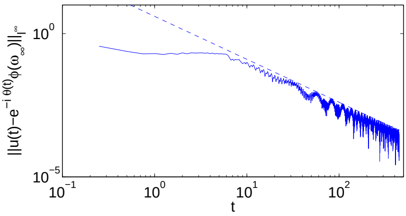
Numerical results

Pointwise estimates imply that $\|\mathbf{z}(t)\|_{l^\infty} = O(t^{-1/3})$ as $t \to \infty$.

Strichartz estimates imply that $\|\mathbf{z}(t)\|_{l^\infty} = O(t^{-1/6+\nu})$, $\nu > 0$ as $t \to \infty$.

For any $p = 1, 2, 3$, it was found that $\|\mathbf{z}(t)\|_{l^\infty} = O(t^{-3/2})$ as $t \to \infty$.





Further directions

- Cuccagna (2009): long-term oscillations of discrete solitons with V supporting two eigenvalues - no proof of existence of the time-periodic space-localized breathers

- Mielke & Patz (2010): better pointwise dispersive decay estimates.

Lemma

For any $q \in [2,4) \cup (4,\infty]$, *there is* $C_q > 0$ *such that*

$$\left\| e^{-itH} P_c \mathbf{f} \right\|_{l^q_n} \quad \leq \quad C_q (1+t)^{-\alpha_q} \|\mathbf{f}\|_{l^1_n},$$

where

$$\alpha_q = \frac{q-2}{2q}, \quad 2 \leq q < 4, \quad \alpha_q = \frac{q-1}{3q}, \quad 4 < q \leq \infty.$$

Scattering to zero solution is proved via standard arguments for

$$i\dot{u}_n = -\Delta u_n + \gamma |u_n|^{2p} u_n = 0, \quad n \in \mathbb{Z},$$

with $p \geq 2$.

Further directions

- P, Sakovich (2010): the proof of the spectral conjecture in the linearization of the discrete soliton for

$$i\dot{u}_n = -\epsilon\Delta u_n + \gamma|u_n|^{2p}u_n = 0, \quad n \in \mathbb{Z},$$

  in the anti-continuum limit $\epsilon \to 0$.

- Ablowitz & Ladik (1975): an integrable version of the cubic DNLS equation

$$i\dot{u}_n + u_{n+1} - 2u_n + u_{n-1} + |u_n|^2(u_{n+1} + u_{n-1}) = 0, \quad n \in \mathbb{Z}.$$

  This equation is related to the Lax operator (the discrete version of the Zakharov–Shabat scattering problem).