

Stability of periodic waves in the NLS model with intensity-dependent dispersion

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Conference on Analysis, Modeling and Numerics of Waves
Delft, The Netherlands, August 26–28, 2006

NLS-IDD model

The NLS equation with intensity-dependent dispersion (IDD):

$$i\psi_t + \alpha(1 - |\psi|^2)\psi_{xx} + \beta|\psi|^2\psi = 0, \quad x \in \mathbb{R}.$$

C.Y. Lin, J.H. Chang, G. Kurizki, and R.K. Lee, Optics Letters **45** (2020) 1471–1474

R.M. Ross, P.G. Kevrekidis, and D.P., Quart. Appl. Math. **79** (2021) 641–665

D.P, R.M. Ross, and P.G. Kevrekidis, J. Phys. A: Math. Theor. **54** (2021) 445701

P.G. Kevrekidis, D.P, and R.M. Ross, IMA J. Appl. Math. **89** (2024) 989–1005

F. Natali, D.P., and S. Wang (2026), arXiv:2603.28514

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D.P, R.M. Ross, and P.G. Kevrekidis, J. Phys. A: Math. Theor. **54** (2021) 445701
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NLS-IDD is a continuum limit of the lattice NLS model:

$$i\partial_\tau \psi_n + (1 - |\psi_n|^2)(\psi_{n+1} + \psi_{n-1}) + \mu|\psi_n|^2\psi_n = 0, \quad n \in \mathbb{Z}$$

with $\mu = 2 + h^2$ and $\psi_n(\tau) = e^{2i\tau}\psi(h^2\tau, hn)$ as $h \rightarrow 0$.

J. Gomez-Gardenes, B. Malomed, L. Floria, A. Bishop, Phys. Rev. E **73** (2006) 036608
J. Adiazola and P. Kevrekidis, Phys. Rev. E **112** (2024) 064207

Another quasilinear NLS equation

The NLS-IDD model is a variant of the quasilinear NLS model:

$$i\psi_t + \alpha\psi_{xx} + \beta\psi|\psi|^2 + \beta\epsilon^2\psi(|\psi|^2)_{xx} = 0, \quad x \in \mathbb{R}.$$

The quasilinear NLS model follows from the nonlocal Kerr model:

$$i\psi_t + \alpha\psi_{xx} + \beta\psi(K_\epsilon * |\psi|^2) = 0,$$

in the asymptotic limit:

$$\hat{K}_\epsilon(k) = 1 - \epsilon^2 k^2 + \mathcal{O}(\epsilon^4), \quad k \in \mathbb{R}.$$

- I. Iliev and K. Kirchev, *Differential Integral Equations* **6** (1993) 685–703
M. Colin, L. Jeanjean, and M. Squassina, *Nonlinearity* **23** (2010) 1353–1385
A. de Laire, and S. López-Martínez, *Calc. Var. PDEs* **64** (2025) 138
A. de Laire and E. Le Quiniou, *Nonlinear Anal.* **265** (2026) 114027

NLS-IDD model in the periodic domain

We consider the NLS-IDD model for L -periodic solutions:

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} + |\psi|^2\psi = 0, \quad x \in [0, L], \quad t > 0.$$

It admits the following conserved quantities:

- energy for $\psi \in H_{\text{per}}^1$ due to the time translation symmetry:

$$E(\psi) = \int_0^L (|\psi_x|^2 + |\psi|^2) dx.$$

- mass for $\psi \in H_{\text{per}}^1$ with small $\|\psi\|_{L^\infty} \leq C < 1$ due to the phase rotation symmetry:

$$Q(\psi) = - \int_0^L \log |1 - |\psi|^2| dx$$

- momentum for $\psi \in H_{\text{per}}^1$ such that $\psi \neq 0$ due to the spatial translation symmetry.

Local well-posedness of quasilinear NLS models

For general quasilinear NLS equations,

$$i\psi_t + f(|\psi|^2)\psi + g(|\psi|^2)\psi_{xx} = 0, \quad x \in \mathbb{R} \text{ or } x \in \mathbb{T},$$

local well-posedness in Sobolev spaces has some history.

- $H^\infty(\mathbb{R})$: M. Poppenberg, Nonlinear Anal. **45** (2001) 723
- $H^s(\mathbb{R})$, for large $s \gg 1$:
C. Kenig, G. Ponce, L. Vega, Inventiones Math. **158** (2004) 343
J. Marzuola, J. Metcalfe, D. Tataru, Kyoto J. Math. **54** (2014) 529
- $H^s(\mathbb{R})$, for $s > 2$ and $s > 1$:
J. Marzuola, J. Metcalfe, D. Tataru, ARMA **242** (2021) 1129
M. Ifrim, D. Tataru, Inventiones Math. **242** (2025) 221
- H_{per}^s , for $s > 2$:
R. Feola, F. Indoli, Ann. I. H. Poincaré **36** (2019) 119
R. Feola, B. Grebert, F. Indoli, Analysis & PDE **16** (2023) 1133

Periodic waves

These are the standing wave solutions of the form $\psi(x, t) = e^{i\omega t}\varphi(x)$ in

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} + |\psi|^2\psi = 0,$$

where φ is a real-valued profile satisfying

$$\frac{d^2\varphi}{dx^2} = \frac{(\omega - \varphi^2)}{(1 - \varphi^2)}\varphi = -\frac{dV}{d\varphi},$$

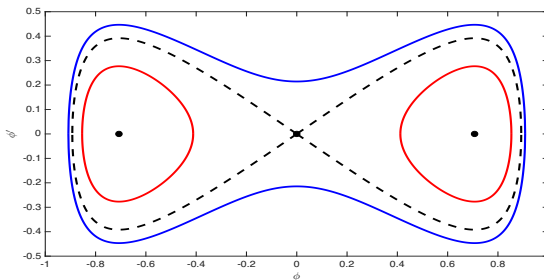
which is integrable as

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad V(\varphi) := \frac{\omega - 1}{2} \log |1 - \varphi^2| - \frac{1}{2}\varphi^2.$$

- $(0, 0)$ is a saddle point if $\omega > 0$,
- $(0, 0)$ is a center point if $\omega < 0$.

Periodic waves

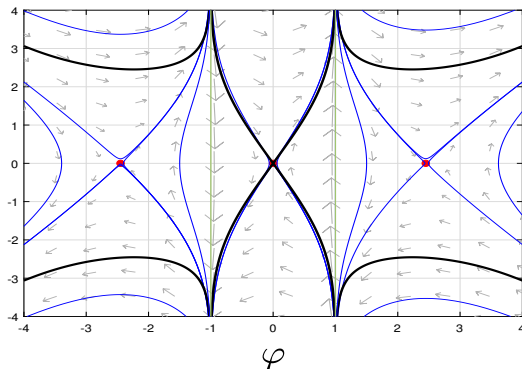
The phase portrait for $\omega \in (0, 1)$ takes the form:



Smooth periodic orbits have different periods for fixed $\omega \in (-\infty, 1)$. For PDE and stability analysis (and applications), we need to find the periodic orbits with different values of the parameter ω for fixed period $L > 0$.

Periodic waves

For $\omega \in (1, \infty)$, all solutions are singular or unbounded:



Existence of periodic waves

Theorem (Natali–P–Wang, 2026)

Fix the spatial period $L > 0$ and define

$$\omega_L = \frac{2\pi^2}{L^2 + 2\pi^2}, \quad \Omega_L = -\frac{4\pi^2}{L^2}.$$

For any $\omega \in (\omega_L, 1)$, there exists a L -periodic **even wave** satisfying

$$\begin{cases} 0 < \varphi(x) < 1, & \forall x \in \mathbb{T}_L, \\ \varphi(x) = \varphi(-x), & \forall x \in \mathbb{T}_L. \end{cases}$$

For any $\omega \in (\Omega_L, 1)$, there exists a L -periodic **odd wave** satisfying

$$\begin{cases} -1 < \varphi(x) < 1, & \forall x \in \mathbb{T}_L, \\ \varphi(x) = -\varphi(-x) = \varphi\left(\frac{L}{2} - x\right), & \forall x \in \mathbb{T}_L. \end{cases}$$

Each generates a 2-parameter orbit $\{\varphi(\cdot - s)e^{-i\theta}\}_{\xi, \theta \in \mathbb{R}}$ due to symmetries.

Existence of periodic waves

Even wave is constructed in

$$H_{\text{per,e}}^2 = \{f \in H_{\text{per}}^2 : f(-x) = f(x) \ x \in \mathbb{T}_L\},$$

by using local bifurcation for zeros of

$$F : H_{\text{per,e}}^2 \times \mathbb{R} \rightarrow L_{\text{per,e}}^2, \quad F(\varphi, \omega) = -(1 - \varphi^2)\varphi'' + \omega\varphi - \varphi^3.$$

near $(\varphi, \omega) = (\sqrt{\omega_L}, \omega_L)$ with $\omega_L = \frac{2\pi^2}{L^2 + 2\pi^2}$ for $\omega > \omega_L$.

Odd wave is constructed in

$$H_{\text{per,o}}^2 = \{f \in H_{\text{per}}^2 : f(-x) = -f(x) \ x \in \mathbb{T}_L\},$$

by using local bifurcation for zeros of $F : H_{\text{per,o}}^2 \times \mathbb{R} \rightarrow L_{\text{per,o}}^2$ near $(\varphi, \omega) = (0, \Omega_L)$ with $\Omega_L = -\frac{4\pi^2}{L^2}$ for $\omega > \Omega_L$.

Existence of periodic waves

To show that the even wave exists for all $\omega \in (\omega_L, 1)$ and the odd wave exists for all $\omega \in (\Omega_L, 1)$, as well as to show continuity and smoothness of the mapping $\omega \rightarrow \varphi \in H_{\text{per}}^2$, we use the period function:

$$T(\mathcal{E}, \omega) = \oint \frac{d\varphi}{\sqrt{2(\mathcal{E} - V(\varphi))}},$$

for each first-order invariant

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad V(\varphi) := \frac{\omega - 1}{2} \log |1 - \varphi^2| - \frac{1}{2}\varphi^2.$$

The period function also helps in the stability analysis of periodic waves.

Period function and its applications

For orbits on the phase plane, we consider the period function

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad T(\mathcal{E}, \omega) = \oint \frac{d\varphi}{\sqrt{2(\mathcal{E} - V(\varphi))}}.$$

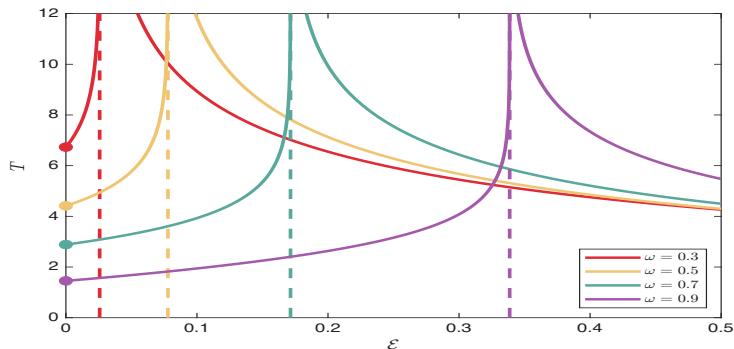


Figure: The period function $T(\mathcal{E}, \omega)$ versus $\mathcal{E}$ for fixed values of ω .

Period function and its applications

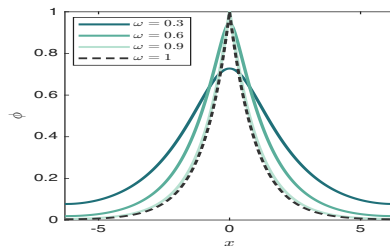
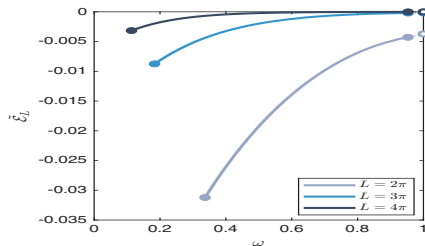
Theorem (Natali–P–Wang, 2026)

The period function $T = T(\mathcal{E}, \omega)$ is a C^1 function of $\mathcal{E} \in (0, \infty) \setminus \mathcal{E}_\omega$ if $\omega \in (0, 1)$ and $\mathcal{E} \in (\mathcal{E}_\omega, \infty)$ if $\omega \in (-\infty, 0)$, where $\mathcal{E}_\omega = V(0)$.

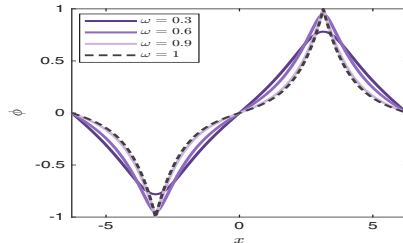
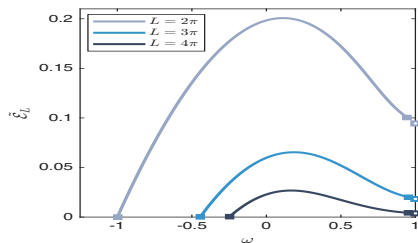
- *For any $\omega \in (0, 1)$, the mapping $(0, \mathcal{E}_\omega) \ni \mathcal{E} \rightarrow T(\mathcal{E}, \omega)$ is monotonically increasing.*
- *For any $\omega \in (-\infty, 1)$, the mapping $(\mathcal{E}_\omega, \infty) \ni \mathcal{E} \rightarrow T(\mathcal{E}, \omega)$ is monotonically decreasing.*

- For any fixed $L > 0$ and $\omega \in (\omega_L, 1)$, there exists a unique root $\mathcal{E}_L(\omega) \in (0, \mathcal{E}_\omega)$ of $T(\mathcal{E}, \omega) = L$ for **the even wave**.
- For any fixed $L > 0$ and $\omega \in (\Omega_L, 1)$, there exists a unique root $\mathcal{E}_L(\omega) \in (\mathcal{E}_\omega, \infty)$ of $T(\mathcal{E}, \omega) = L$ for **the odd wave**.

Even waves:



Odd waves:



From smooth profiles to peaked profiles

Both families of solutions terminate at $\omega = 1$, for which

$$\frac{d^2\varphi}{dx^2} = \frac{(\omega - \varphi^2)}{(1 - \varphi^2)}\varphi$$

becomes the linear equation $\varphi''(x) = \varphi(x)$ with the exact solution:

$$\omega = 1 : \quad \varphi(x) = \frac{\cosh\left(\frac{L}{2} - |x|\right)}{\cosh\left(\frac{L}{2}\right)}, \quad \mathcal{E}_L = -\frac{1}{2 \cosh^2\left(\frac{L}{2}\right)}$$

and a similar peaked solution for the odd wave with

$$\omega = 1 : \quad \mathcal{E}_L = \frac{1}{2 \sinh^2\left(\frac{L}{2}\right)}.$$

Remark: Numerical detection of roots is a stiff problem near $\omega = 1$. The numerical data is missing in a tiny gap near $\omega = 1$.

Monotonicity of the period function

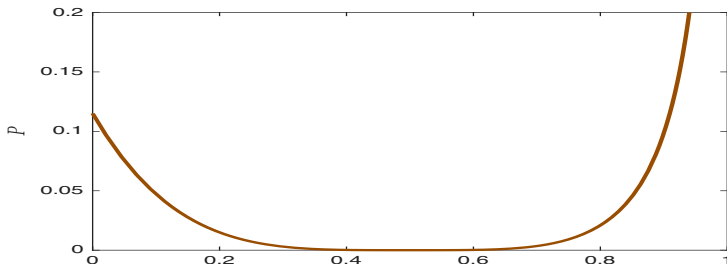
Even waves: By a theorem of C. Chiconi (1987), the period function

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = \mathcal{E}, \quad T(\mathcal{E}, \omega) = 2 \int_m^M \frac{d\varphi}{\sqrt{2(\mathcal{E} - V(\varphi))}}, \quad 0 < m < M < 1,$$

is monotonically increasing in $\mathcal{E}$ if $I''(\varphi) > 0$, $\varphi \in (0, 1)$, where

$$I(\varphi) = \frac{V(\varphi)}{[V'(\varphi)]^2} \quad \Rightarrow \quad I''(\varphi) = \frac{P(\varphi)}{(V'(\varphi))^4(1 - \varphi^2)^4},$$

where $P(\varphi)$ is a combination of polynomials and logarithmic functions.



Monotonicity of the period function

Odd waves: The period function can be transformed at $\mathcal{E} = V(M)$:

$$\begin{aligned} T(\mathcal{E}, \omega) &= 4 \int_0^M \frac{d\varphi}{\sqrt{2[V(M) - V(\varphi)]}} \\ &= \sqrt{2} \int_0^q \frac{dt}{\sqrt{t[W(q) - W(t)]}} \\ &= \int_0^1 \frac{\sqrt{2q}}{\sqrt{u[W(q) - W(qu)]}} du, \end{aligned}$$

where $q = M^2$, $t = \varphi^2 = qu$, and

$$W(t) := \frac{1}{2}(\omega - t) + \frac{1}{2}(1 - \omega) \log \frac{1 - \omega}{1 - t}, \quad t \in (0, 1).$$

Equipped with the analysis of convexity, this yields

$$\frac{\partial T}{\partial q} = \frac{1}{\sqrt{2q}} \int_0^1 \frac{du}{\sqrt{u[W(q) - W(qu)]}} - \frac{\sqrt{q}}{\sqrt{2}} \int_0^1 \frac{W'(q) - uW'(qu)}{\sqrt{u[W(q) - W(qu)]}^3} du < 0.$$

Energetic stability of the even wave

Theorem (Natali–P–Wang, 2026)

Fix the spatial period $L > 0$. The profile $\varphi \in H_{\text{per}}^1$ is a C^1 function of ω for the even wave in $(\omega_L, 1)$. For any $\omega \in (\omega_L, 1)$, the even wave is a local minimizer of energy E for a fixed mass Q in H_{per}^1 , which is degenerate only due to translational and rotational symmetries, if and only if the mapping $\omega \rightarrow Q(\varphi)$ is monotonically increasing.

Energetic stability of the even wave

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Remark:

This result yields spectral stability of even wave. It does not imply the orbital stability along the orbit $\{\varphi(\cdot - \xi)e^{i\theta}\}_{\xi, \theta \in \mathbb{R}}$ because of the lack of local well-posedness in H_{per}^1 . We call it the energetic stability.

Analysis of energetic stability

Recall the conserved quantities

$$E(\psi) = \oint (|\psi_x|^2 + |\psi|^2) dx, \quad Q(\psi) = - \oint \log |1 - |\psi|^2| dx.$$

The Hamiltonian structure of the NLS equation is non-standard:

$$i\partial_t \psi = (1 - |\psi|^2) \frac{\delta}{\delta \bar{\psi}} (E - Q), \quad \frac{\delta}{\delta \bar{\psi}} (E - Q) = -\partial_x^2 \psi - \frac{|\psi|^2 \psi}{1 - |\psi|^2}.$$

Conservation of Q is related to the phase rotation symmetry due to

$$\psi = (1 - |\psi|^2) \frac{\delta Q}{\delta \bar{\psi}}.$$

The standing wave $\psi(x, t) = e^{i\omega t} \varphi_\omega$ is defined by a critical point of $\Lambda_\omega := E + (\omega - 1)Q$ with

$$0 = (1 - |\varphi|^2) \frac{\delta}{\delta \bar{\varphi}} (E + (\omega - 1)Q).$$

Expansion of $\Lambda_\omega = E + (\omega - 1)Q$ near the periodic profile φ yields

$$\Lambda_\omega(\varphi + u + iv) - \Lambda_\omega(\varphi) = \langle \mathcal{S}_+ u, u \rangle_{L^2} + \langle \mathcal{S}_- v, v \rangle_{L^2} + \mathcal{O}(\|u + iv\|_{H_{\text{per}}^1}^3),$$

where $\mathcal{S}_\pm : H_{\text{per}}^2 \subset L_{\text{per}}^2 \rightarrow L_{\text{per}}^2$ given by

$$\mathcal{S}_- = -\partial_x^2 + \frac{\omega - \varphi^2}{1 - \varphi^2}, \quad \mathcal{S}_+ = -\partial_x^2 + \frac{\omega + 2\varphi\partial_x^2\varphi - 3\varphi^2}{1 - \varphi^2}.$$

Sturm's theory for periodic Schrödinger operators

Spectrum of either $\mathcal{S}_+$ or $\mathcal{S}_-$ in L^2_{per} consists of an unbounded sequence of real eigenvalues satisfying

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 \dots < \lambda_{2n-1} \leq \lambda_{2n} \dots,$$

where the eigenfunction for λ_{2n-1} and λ_{2n} has exactly $2n$ zeroes in the periodic domain.

Let $n(\mathcal{S}_{\pm})$ denote the Morse index (number of negative eigenvalues) and $z(\mathcal{S}_{\pm})$ denote multiplicity of the zero eigenvalue.

- Since $\mathcal{S}_-\varphi = 0$ and $\varphi > 0$, then $\lambda_0 = 0$ and $n(\mathcal{S}_-) = 0$, $z(\mathcal{S}_-) = 1$.
- Since $\mathcal{S}_+\partial_x\varphi = 0$ and $\partial_x\varphi$ has exactly two nodes, then either $\lambda_1 = 0 < \lambda_2$, or $\lambda_1 < \lambda_2 = 0$, or $\lambda_1 = \lambda_2 = 0$. However, this question is completely solved with $\partial_{\varepsilon}T > 0$, or $\partial_{\varepsilon}T < 0$, or $\partial_{\varepsilon}T = 0$. Due to strict increase of the period function, we have $n(\mathcal{S}_+) = z(\mathcal{S}_+) = 1$.

Constrained minimization of $\Lambda_\omega = E + (\omega - 1)Q$

Due to non-degeneracy

$$\text{Ker}(\mathcal{S}_+) = \text{span}(\partial_x \varphi), \quad \text{Ker}(\mathcal{S}_-) = \text{span}(\varphi),$$

the mapping $\omega \rightarrow \varphi \in H_{\text{per}}^1$ is C^1 and

$$\mathcal{S}_+ \partial_\omega \varphi = -\frac{\varphi}{1 - \varphi^2} = \frac{\delta Q}{\delta \bar{\varphi}}.$$

The saddle point of $\Lambda_\omega = E + (\omega - 1)Q$ is a strict (local) minimizer of E subject to fixed Q and the two symmetries if and only if

$$\langle \mathcal{S}_+^{-1} \frac{\varphi}{1 - \varphi^2}, \frac{\varphi}{1 - \varphi^2} \rangle_{L_{\text{per}}^2} = -\langle \frac{\varphi}{1 - \varphi^2}, \partial_\omega \varphi \rangle_{L_{\text{per}}^2} = -\frac{1}{2} \frac{d}{d\omega} Q(\varphi) < 0.$$

A twist for the spectral stability argument

The energetic stability implies that the linear spectral problem

$$\begin{pmatrix} 0 & \mathcal{S}_- \\ -\mathcal{S}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

admits no eigenvalues $\lambda \in \mathbb{C} \setminus \{i\mathbb{R}\}$ with $(u, v) \in H_{\text{per}}^2 \times H_{\text{per}}^2$.

However, the correct spectral stability problem is

$$\begin{pmatrix} 0 & (1 - \varphi_\omega^2)\mathcal{S}_- \\ -(1 - \varphi_\omega^2)\mathcal{S}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

defined in the weighted Hilbert space $\mathcal{H} \times \mathcal{H}$,
 $\mathcal{H} := L^2([0, L], (1 - \varphi_\omega^2)^{-1} dx)$.

However, smooth profiles satisfy $0 \leq \varphi_\omega(x) < 1$ and hence $(1 - \varphi_\omega^2)^{1/2}$ is bounded and invertible.

By using

$$u = (1 - \varphi_\omega^2)^{1/2} \tilde{u}, \quad v = (1 - \varphi_\omega^2)^{1/2} \tilde{v},$$

the correct spectral stability problem

$$\begin{pmatrix} 0 & (1 - \varphi_\omega^2) \mathcal{S}_- \\ -(1 - \varphi_\omega^2) \mathcal{S}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

becomes

$$\begin{pmatrix} 0 & (1 - \varphi_\omega^2)^{1/2} \mathcal{S}_- (1 - \varphi_\omega^2)^{1/2} \\ -(1 - \varphi_\omega^2)^{1/2} \mathcal{S}_+ (1 - \varphi_\omega^2)^{1/2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix},$$

where $(1 - \varphi_\omega^2)^{1/2} \mathcal{S}_\pm (1 - \varphi_\omega^2)^{1/2}$ have the same number of negative and zero eigenvalues as $\mathcal{S}_\pm$ in $L^2(\mathbb{R})$ by Sylvester's inertia law.

This trick yields **spectral stability** of the even wave from their **energetic stability**.

Energetic stability of the odd wave

Theorem (Natali–P–Wang, 2026)

Fix the spatial period $L > 0$. The profile $\varphi \in H_{\text{per}}^1$ is a C^1 function of ω for the odd wave in $(\Omega_L, 1)$. For any $\omega \in (\Omega_L, 1)$, the odd wave is a local minimizer of energy E for a fixed mass Q in $\mathcal{Y} \subset H_{\text{per}}^1$, which is degenerate only due to translational and rotational symmetries, if and only if the mapping $\omega \rightarrow Q(\varphi)$ is monotonically increasing, where

$$\mathcal{Y} = \left\{ u \in H_{\text{per}}^1 : u\left(\frac{L}{2} - x\right) = -u\left(x - \frac{L}{2}\right), \quad \forall x \in \mathbb{T}_L \right\}.$$

- Since $\mathcal{S}_-\varphi = 0$ and φ has two nodes, then $n(\mathcal{S}_-) = z(\mathcal{S}_-) = 1$.
- Since $\mathcal{S}_+\partial_x\varphi = 0$ and $\partial_x\varphi$ has also two nodes, then the strict decrease of the period function implies $n(\mathcal{S}_+) = 2$ and $z(\mathcal{S}_+) = 1$.

The extra symmetry was used before, e.g. in T.Gallay–M.Haragus (2007).

Criterion for the energetic stability

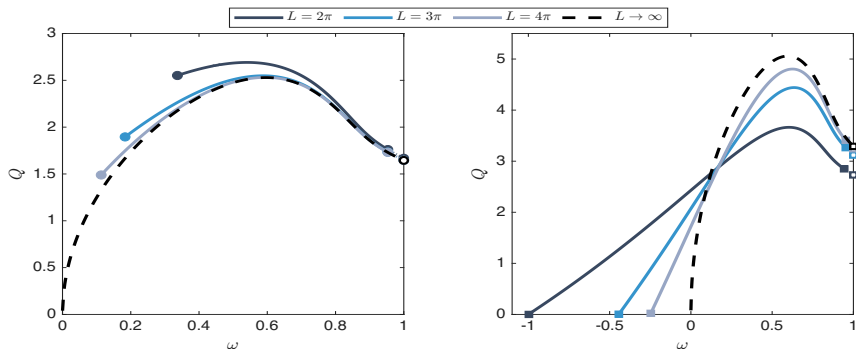
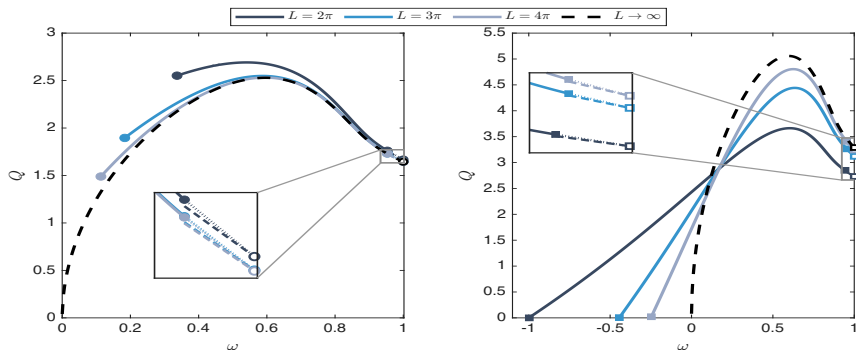


Figure: Dependence of $Q(\varphi)$ versus ω for $L = 2\pi, 3\pi, 4\pi$ and in the limit $L \rightarrow \infty$.

Conjecture: There exist $\omega_*, \Omega_* \in (0, 1)$ such that the even wave is energetically stable for $\omega \in (\omega_L, \omega_*)$ and unstable for $\omega \in (\omega_*, 1)$, whereas the odd wave is stable for $\omega \in (\Omega_L, \Omega_*)$ and unstable for $\omega \in (\Omega_*, 1)$.

Very recent addition to the proof of the conjecture

S. Wang computed very precise asymptotics of $Q(\varphi)$ near the local bifurcation at $\omega = \omega_L, \Omega_L$ and near the peaked limit $\omega = 1$.



Numerical explorations from Kevrekidis–P–Ross (2024)

A bubble of instability is detected for $\omega \in (\omega_1, \omega_2)$, $0 < \omega_1 < \omega_2 < 1$.

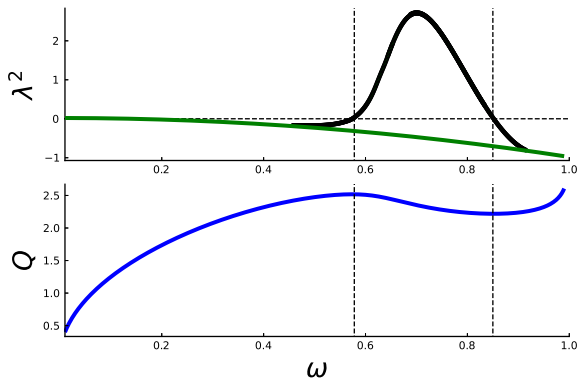


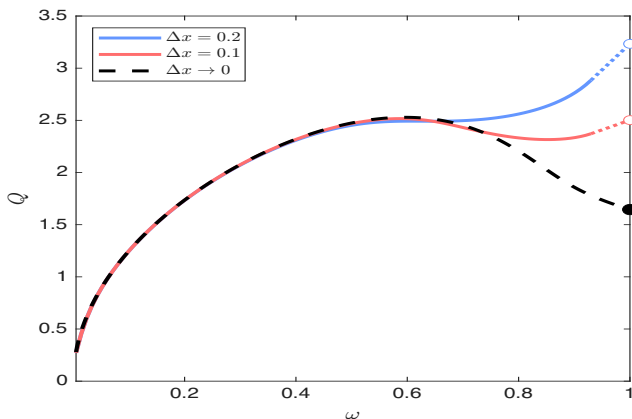
Figure: Top: Squared eigenvalue λ^2 and the map $\omega \mapsto Q(\varphi_\omega)$.

Transition at ω_1 and ω_2 between unstable and stable branches is consistent between two methods. Yet, the second transition is a numerical artefact.

Numerical explorations in Natali–P–Wang (2026)

By using the finite-difference approximation, we have recovered the numerical artefact with “stable slope” of the mapping $\omega \rightarrow Q(\varphi)$.

$$(1 - \varphi_j^2)(D^2\varphi)_j - (\omega - \varphi_j^2)\varphi_j = 0, \quad (D^2\varphi)_j = \frac{\varphi_{j-1} - 2\varphi_j + \varphi_{j+1}}{(\Delta x)^2}.$$



Conclusion

- We considered new variations of the cubic NLS model with the intensity-dependent dispersion.
- We proved spectral and energetic stability of the periodic waves with precise proofs of monotonicity of the period function and a nearly complete proof of the slope of the mass function.
- Local well-posedness of the quasilinear NLS models with the intensity-dependent dispersion in the energy space is still open.

THE END!