The NLS-BO equation arising in the continuum approximation of the Calogero-Moser-Sutherland model

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Section 1. The BO and NLS-BO models

The Benjamin–Ono equation is a model for long internal waves in deep fluid:

$$u_t + uu_x + H(u_{xx}) = 0, \quad H(u) := \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{u(y,t)dy}{y-x}. \tag{BO}$$

B.T. Benjamin, J. Fluid Mech. **29** (1967) 559

H. Ono, J. Phys. Soc. Japan **39** (1975) 1082–1091.

Question: Given a slow modulation of the linear waves,

$$u(x,t) = \varepsilon A(\varepsilon(x-2|k|t), \varepsilon^2 t)e^{ikx-ik|k|t} + \varepsilon \bar{A}(\varepsilon(x-2|k|t), \varepsilon^2 t)e^{-ikx+ik|k|t} + \mathcal{O}(\varepsilon^2),$$

what is the normal form for the complex amplitude $A = A(\xi, \tau)$?

For the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, the answer to this question is the defocusing cubic NLS equation

$$iA_\tau + A_{\xi\xi} - |A|^2 A = 0, \tag{NLS}$$

where the constant-amplitude background is linearly and nonlinearly stable and stable dark solitons propagate at the constant background.

Modulation equation for the BO equation

For the BO equation, the dispersion relation $\omega(k) = k|k|$ yields $\omega''(k) = 2$ for $k > 0$ so that the linear Schrödinger equation $iA_\tau + A_{\xi\xi} = 0$ holds for linear perturbations. However, the coefficient of the cubic term $|A|^2 A$ is *zero* and the derivative NLS equation was obtained as

$$iA_\tau + A_{\xi\xi} + iA(|A|^2)_\xi = 0.$$

M. Tanaka, J. Phys. Soc. Japan **51** (1982) 2686

However, the mean field was computed incorrectly in the derivation of the cubic derivative term. With the account of the mean field and the asymptotic multi-scale expansions, the correct modulation equation was obtained in the form:

$$iA_\tau + A_{\xi\xi} + A(i + H)(|A|^2)_\xi = 0. \qquad \text{(NLS-BO+)}$$

D. Pelinovsky, Phys. Lett. A **197** (1995) 401-406
R. Grimshaw, D. Pelinovsky, J. Math. Phys. **36** (1995) 4203–4219

V. Hur, M. Johnson, SIAM J. Math. Anal. **47** (2015) 3528–3554.

Similarly to the defocusing NLS, the constant-amplitude solutions of NLS-BO+ are linearly stable and stable dark solitons propagate on the constant background.

Y. Matsuno, Phys. Lett. A **278** (2000) 53-58

Y. Matsuno, J. Phys. Soc. Japan **73** (2004) 3285–3293

Dynamics of poles of BO as particles

The BO equation admits the meromorphic solutions in the form

$$u(x,t) = i\sum_{j=1}^{N}\left(\frac{1}{x-x_j(t)} - \frac{1}{x-\bar{x}_j(t)}\right), \quad \mathrm{Im}\, x_j(t) < 0, \quad \forall j,$$

where $\{x_j(t)\}_{j=1}^{N}$ satisfy the Calogero–Moser (CM) system:

$$\frac{d^2 x_j}{dt^2} = 8\sum_{k\neq j}\frac{1}{(x_j - x_k)^3}.$$

The CM system can be generalized with the inverse squared elliptic potentials, where it is usually referred to as the Calogero–Moser–Sutherland (CMS) system. For the elliptic potentials degenerated into the trigonometric potentials, it is

$$\frac{d^2 x_j}{dt^2} = -\frac{\partial}{\partial x_j}\sum_{k\neq j}\left[\frac{\pi}{L}.\cot\frac{\pi}{L}(x_j - x_k)\right]^2.$$

F. Calogero, J. Math. Phys. **12** (1971); B. Sutherland, Phys. Rev. A **4** (1971); J. Moser, Adv. Math. **16** (1975)

Question: What is the continuum limit of the CMS system?

B. Ingimarson, R.L. Pego, SIAM J. Appl. Math. **84** (2024); B. Ingimarson, R.L. Pego, Nonlinearity **37** (2025) 125016;

J. D. Wright, SIAM J. Math. Anal. **56** (2024) 5583; U. Akpan, J.D. Wright, Stud. Appl. Math. **155** (2025) e70119

Another version of the NLS-BO equation

By using the continuous approximation, a "bi-directional" BO equation was obtained in A. Abanov, E. Betterlheim, and P. Wiegmann, J. Phys. A: Math. Theor. **42** (2009) 135201 which defines the hydrodynamic form of another version of the NLS–BO equation:

$$iA_\tau + A_{\xi\xi} - A(i + H)(|A|^2)_\xi = 0. \qquad \text{(NLS-BO-)}$$

This "focusing" NLS–BO equation has recently picked interest:

- Well-posedness, bright solitons, and blow-up:

  P. Gérard, E. Lenzmann, Comm. Pure Appl. Math. **77** (2024) 4008–4062
  Y. Matsuno, Stud. Appl. Math. **151** (2023) 883-922
  J. Hogan, M. Kowalski, Pure Appl. Anal. **6** (2024) 941-–954
  R. Killip, T. Laurens, M. Visan, Commun. AMS **5** (2025), 284–320

  K. Kim, T. Kim, S. Kwon, arXiv:2404.09603, arXiv:2408.12843, arXiv:2412.12518

- Periodic traveling waves and Lax spectrum:

  R. Badreddine, Pure Appl. Anal. **6** (2023) 379-414

  R. Badreddine, Ann. Inst. H. Poincaré C Anal. Non Linéaire **42** (2025) 1037–1092

- Coupled integrable systems of the NLS-BO-type:

  B. K. Berntson, A. Fagerlund, Physica D **451** (2024) 133762

  R. Sun, Lett. Math. Phys. **114** (2024) 74

Our contribution

For both versions of the NLS–BO equation

$$iu_t = u_{xx} \pm u(i + H)(|u|^2)_x \qquad \text{(NLS-BO}\pm\text{)}$$

we study

- if the constant background is stable linearly and nonlinearly;
- what is the Lax spectrum for traveling periodic waves;
- if families of (bright) and (dark) breathers exist on the periodic wave;
- if modulational instability and rogue waves exist in the focusing case.

J. Chen, D. Pelinovsky, Nonlinearity **38** (2025) 075016

Section 2. Stability of the constant background

For the NLS–BO equation,

$$iu_t = u_{xx} + \sigma u(i + H)(|u|^2)_x, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad \sigma = \pm 1,$$

the main question is whether the focusing $(\sigma = -1)$ and defocusing $(\sigma = +1)$ cases have different conclusions in the modulational stability of the constant solution as it happens for the local NLS equation

$$iu_t = -u_{xx} + \sigma |u|^2 u, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad \sigma = \pm 1.$$

Due to symmetries:

$$u(x,t) \mapsto u(x + x_0, t + t_0)e^{i\theta_0}, \quad u(x,t) \mapsto \alpha u(\alpha^2 x, \alpha^4 t), \quad x_0, t_0, \theta_0, \alpha \in \mathbb{R},$$

it is sufficient to normalize the background to unity: $u = 1$.

Linear stability theorem

Theorem 1 (J. Chen & D.P., 2025). *Let $u = 1 + v$ and consider the linearized equations of motion*

$$iv_t = v_{xx} + \sigma(i + H)(v_x + \bar{v}_x).$$

In the defocusing case $\sigma = +1$, for every initial data $v_0 \in H^s(\mathbb{R})$, $s \geq 0$, the unique solution $v \in C(\mathbb{R}, H^s(\mathbb{R}))$ with $v|_{t=0} = v_0$ satisfies

$$\|v(\cdot, t)\|_{H^s} \leq C\|v_0\|_{H^s} \quad \textit{for every} \quad t \in \mathbb{R},$$

for some constant $C > 0$.

Remark 2. *In the focusing case $\sigma = -1$, there is a resonance of two Fourier modes which suggests the linear instability of the constant solution $u = 1$ in the space of 2π-periodic functions. There is no linear intability in $L^2_{\text{per}}(0, T)$ if the spatial period T is not divisible by 2π and in $L^2(\mathbb{R})$ if the Fourier transform of $v|_{t=0} = v_0$ is zero at the two resonant Fourier modes.*

Proof of the linear stability (defocusing case)

Separating the real and imaginary parts as $v = A + iB$ and using Fourier transform in x with Fourier parameter $k \in \mathbb{R}$ yields the system

$$\left\{ \begin{array}{l} \hat{A}_t = -k^2 \hat{B} + 2\sigma i k \hat{A}, \\ \hat{B}_t = k^2 \hat{A} + 2\sigma |k| \hat{A}, \end{array} \right.$$

from which we obtain the characteristic equation,

$$\lambda^2 - 2i\sigma k \lambda + k^2 (k^2 + 2\sigma |k|) = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \lambda_1(k) = -ik|k|, \\ \lambda_2(k) = ik(2\sigma + |k|). \end{array} \right.$$

For $\sigma = 1$, there is no resonance $\lambda_1(k) = \lambda_2(k)$ with $k \neq 0$ so that there exists $C > 0$ such that

$$|\hat{A}(k,t)| + |\hat{B}(k,t)| \leq C \left(|\hat{A}(k,0)| + |\hat{B}(k,0)| \right), \qquad t \in \mathbb{R},$$

which implies

$$\|v(\cdot,t)\|_{H^s} \leq C \|v_0\|_{H^s} \quad \text{for every} \quad t \in \mathbb{R}.$$

Proof of the linear stability (focusing case)

For $\sigma = -1$, there is resonance $\lambda_1(k) = \lambda_2(k)$ at $k = \pm 1$ with linear growth of $\hat{A}(\pm 1, t)$ and $\hat{B}(\pm 1, t)$:

$$\left\{ \begin{array}{l} \hat{A}(\pm 1, t) = (\hat{c}_1 + \hat{c}_2 t)\, e^{\mp it}, \\ \hat{B}(\pm 1, t) = (\mp i \hat{c}_1 + (\mp it - 1)\hat{c}_2)\, e^{\mp it}, \end{array} \right.$$

For every $k \in \mathbb{R} \backslash \{+1, -1\}$, we get the solution

$$\left\{ \begin{array}{l} \hat{A}(k,t) = k\hat{C}_1(k) e^{-ik|k|t} + \hat{C}_2(k) e^{ik(|k|-2)t}, \\ \hat{B}(k,t) = i(|k|-2)\hat{C}_1(k) e^{-ik|k|t} - i\mathrm{sgn}(k)\hat{C}_2(k) e^{ik(|k|-2)t}, \end{array} \right.$$

where

$$\hat{C}_1(k) = \frac{i\mathrm{sgn}(k)\hat{A}(k,0) + \hat{B}(k,0)}{2i(|k|-1)}, \qquad \hat{C}_1(k) = \frac{i(|k|-2)\hat{A}(k,0) + \mathrm{sgn}(k)\hat{B}(k,0)}{2i(|k|-1)},$$

which implies

$$\|v(\cdot,t)\|_{H^s} \leq C\|v_0\|_{H^s \cap L^{2,2}}, \quad \text{for every} \quad t \in \mathbb{R}$$

only if $\hat{A}(k,0)$, $\hat{B}(k,0)$ are C^1 at $k = \pm 1$ with $\hat{A}(\pm 1, 0) = \hat{B}(\pm 1, 0) = 0$.

Nonlinear stability theorem (defocusing case)

Theorem 3 (J. Chen & D.P., 2025). *For every fixed* $T > 0$, *there exists* $\delta > 0$ *such that for every* $v_0 \in H^1_{\rm per}((0,T),\mathbb{C})$ *with* $\|v_0\|_{H^1_{\rm per}} \leq \delta$, *the unique solution* $u \in C(\mathbb{R}, H^1_{\rm per}((0,T),\mathbb{C}))$ *with* $u|_{t=0} = 1 + v_0$ *satisfies*

$$\|e^{-i\theta(t)} u(\cdot,t) - 1\|_{H^1_{\rm per}} \leq C \|v_0\|_{H^1_{\rm per}} \quad \textit{for every} \quad t \in \mathbb{R},$$

for some constant $C > 0$ *and some function* $\theta \in C(\mathbb{R})$.

- Local well-posedness in $H^1(\mathbb{R})$ was proven in

R. Moura, D. Pilod, Adv. Diff. Eqs. **15** (2010) 925–952

- There exist infinitely many conserved quantities:

$$I_1(u) = \oint (|u|^2 - 1) dx,$$

$$I_2(u) = i \oint (u\bar{u}_x - \bar{u} u_x) dx + \sigma \oint (|u|^4 - 1) dx,$$

$$I_3(u) = \oint \left(|u_x|^2 - \frac{i}{2}\sigma |u|^2 (\bar{u} u_x - \bar{u}_x u) - \frac{1}{2}\sigma |u|^2 H(|u|^2)_x + \frac{1}{3}(|u|^6 - 1) \right) dx.$$

R. Grimshaw, D. Pelinovsky, J. Math. Phys. **36** (1995) 4203–4219

Lyapunov functional

Combining three conserved quantities together yields the Lyapunov functional

$$\begin{aligned}\Lambda(v) &:= I_3(1+v) - \sigma I_2(1+v) + I_1(1+v) \\ &= \oint \left(|v_x|^2 + \frac{1}{2}\sigma(v+\bar{v})|\partial_x|(v+\bar{v}) + N(v) \right) dx,\end{aligned}$$

where $|\partial_x|$ is a self-adjoint, positive operator in L^2_{per} with $\mathrm{Dom}(|\partial_x|) = H^1_{\mathrm{per}}$.

If $\sigma = +1$ (defocusing case), then

$$\begin{aligned}\oint \left[|v_x|^2 + \frac{1}{2}(v+\bar{v})|\partial_x|(v+\bar{v}) \right] dx &= \sum_{n\in\mathbb{Z}} \frac{4\pi^2 n^2}{T} |\hat{v}_n|^2 + \pi|n||\hat{v}_n + \bar{\hat{v}}_{-n}|^2 \\ &\geq \frac{1}{2}\|v_x\|^2_{L^2} + \frac{2\pi^2}{T^2}\|v - \hat{v}_0\|^2_{L^2}.\end{aligned}$$

If $\hat{v}_0(t)$ is controlled, then coercivity and Banach algebra of H^1_{per} for $N(v)$ yields

$$\begin{aligned}\|v(\cdot,t)\|_{H^1_{\mathrm{per}}} &\leq \|\hat{v}_0(t)\|_{L^2_{\mathrm{per}}} + \|v(\cdot,t) - \hat{v}_0(t)\|_{H^1_{\mathrm{per}}} \\ &\leq \sqrt{T}|\hat{v}_0(t)| + C\sqrt{\Lambda(v)} \leq C\|v_0\|_{H^1_{\mathrm{per}}}.\end{aligned}$$

Control of the mean-value term

$\mathrm{Re}(\hat{v}_0)(t)$ can be controlled from conservation of

$$I_1 = \oint (v + \bar{v} + |v|^2) dx \quad \Rightarrow \quad |\mathrm{Re}(\hat{v}_0)(t)| \leq \frac{\sqrt{T + I_1}}{\sqrt{T}} - 1 \leq C\|v_0\|_{L^2},$$

We also set $\mathrm{Im}(\hat{v}_0)(t) = 0$ by using the orthogonal decomposition

$$u(x,t) = e^{i\theta(t)} \left[1 + v(x,t) \right], \qquad \oint \mathrm{Im}(v)(x,t) dx = 0.$$

The latter is achieved with the implicit function theorem for $f(\theta) : \mathbb{R} \to \mathbb{R}$,

$$f(\theta) = \oint \mathrm{Im}(e^{-i\theta} u - 1) dx$$

in a ball with small $\inf_{\theta \in \mathbb{R}} \|e^{-i\theta} u - 1\|_{H^1_{\mathrm{per}}} \leq C\|v_0\|_{H^1_{\mathrm{per}}}$.

Nonlinear stability (focusing case)

The quadratic part is coercive if $T < \pi$:

$$\oint \left[|v_x|^2 - \frac{1}{2}(v+\bar{v})|\partial_x|(v+\bar{v}) \right] dx = \sum_{n\in\mathbb{Z}} \frac{4\pi^2 n^2}{T} (|\widehat{\mathrm{Re}(v)}_n|^2 + |\widehat{\mathrm{Im}(v)}_n|^2) - 4\pi |n| |\widehat{\mathrm{Re}(v)}|^2,$$

and the orbital stability holds for $T < \pi$ by the same method.

The resonance and the linear instability occurs in L^2_{per} if $T = 2\pi$ or if T is multiple to 2π.

Nonlinear stability of the constant background in the focusing case is an open problem in $H^1_{\mathrm{per}}(0,T)$ with $T \geq \pi$.

Section 3. Periodic waves and their Lax spectrum

The NLS–BO equation,

$$iu_t = u_{xx} + \sigma u(i + H)(|u|^2)_x, \quad \sigma = \pm 1.$$

can be rewritten in terms of the projection operator $\mathcal{P}^+$:

$$iu_t = u_{xx} + 2i\sigma u \mathcal{P}^+(|u|^2)_x.$$

If $f \in L^2_{\rm per}((0,T),\mathbb{C})$ is T-periodic, then Fourier series

$$f = \sum_{n\in\mathbb{Z}} f_n e^{\frac{2\pi i n x}{L}} \quad \Rightarrow \quad \mathcal{P}^+ f = \sum_{n=0}^{\infty} f_n e^{\frac{2\pi i n x}{L}},$$

which is analytically continued in $\mathbb{C}^+$ with a bounded limit as ${\rm Im}(x) \to +\infty$.

Solutions to the NLS–BO equation can be defined in $H^1_{\rm per}$. However, traveling waves and exact solutions are defined in $H^1_{\rm per} \cap L^2_+ = \{u \in H^1_{\rm per} : \; \mathcal{P}^+ u = u\}$.

Exact solutions for the traveling periodic waves

Traveling periodic waves are expressed in elementary functions:

$$u = \frac{g}{f^+}, \quad \bar{u} = \frac{h}{f^-}, \quad |u(x,t)|^2 = 1 - i\sigma \frac{\partial}{\partial x} \ln \frac{f^+}{f^-} = 1 - \frac{\sigma k \sinh\phi}{\cos k(x-ct) + \cosh\phi},$$

with

$$f^\pm = 1 + e^{ik(x-ct)\mp\phi},$$
$$g = e^{\frac{1}{2}(\psi-\phi)}(1 + e^{ik(x-ct)-\psi}),$$
$$h = e^{-\frac{1}{2}(\psi-\phi)}(1 + e^{ik(x-ct)+\psi}),$$

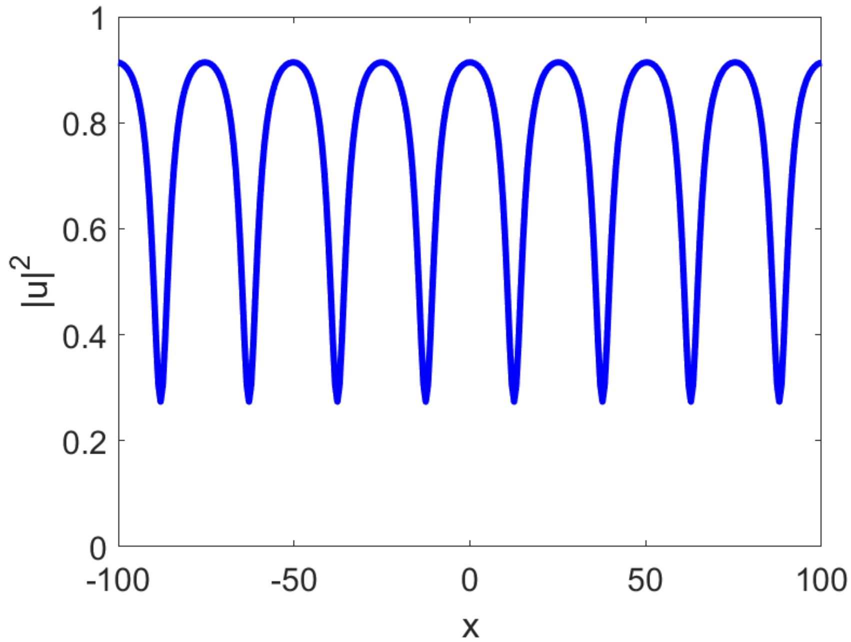
where $k > 0$ and $c \in \mathbb{R}$ are arbitrary parameters satisfying

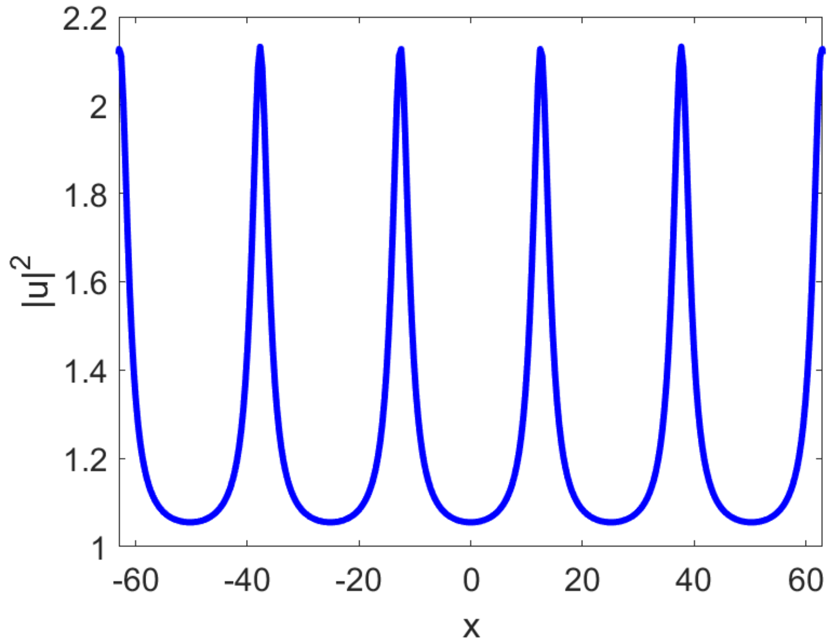
$$e^{2\phi} = \frac{(c-k)(c+k+2\sigma)}{(c+k)(c-k+2\sigma)} > 1, \qquad e^{\psi-\phi} = \frac{c+k}{c-k} > 0.$$

- If $\sigma = 1$, then $k \in (0,1)$ and $c \in (-2+k, -k)$.
- If $\sigma = -1$, then $k \in (0,\infty)$ and either $c \in (k+2,\infty)$ or $c \in (-\infty,-k)$.

Illustrations of periodic waves

Left: defocusing case $\sigma = 1$. Right: focusing case $\sigma = -1$.





The background amplitude can be reduced to 0 in the focusing case.

Lax spectrum for the traveling periodic waves

Solutions of the NLS-BO is the compatibility condition for the linear system:

$$\begin{cases} ip_x + \lambda p + uq^+ = 0, \\ q^+ - q^- + \sigma \bar{u} p = 0, \\ ip_t + \lambda^2 p + \lambda u q^+ + i(uq_x^+ - u_x q^+) = 0, \\ iq_t^\pm - 2i\lambda q_x^\pm + q_{xx}^\pm \pm 2i\sigma q^\pm \mathcal{P}^\pm(|u|^2)_x = 0, \end{cases}$$

where λ is the spectral parameter, and $q^\pm \in L^2_\pm$.

Hence $q^+ = -\sigma \mathcal{P}^+(\bar{u}p)$ and the Lax spectrum is defined by the self-adjoint operator in $\mathcal{L} : H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ given by

$$\mathcal{L} := -i\partial_x + \sigma u \mathcal{P}^+(\bar{u}\,\cdot)$$

If $u \in H^1 \cap L^2_+$, then $p \in H^1 \cap L^2_+$, that is, the Lax spectrum is the set of admissible values of $\mathcal{L}|_{L^2_+}$ given by

$$\mathcal{L}_u|_{L^2_+} = -i\partial_x + \sigma \mathcal{P}^+ u \mathcal{P}^+(\bar{u}\,\cdot)$$

P. Gérard, E. Lenzmann, Comm. Pure Appl. Math. **77** (2024) 4008–4062

R. Badreddine, Ann. Inst. H. Poincaré C Anal. Non Linéaire **42** (2025) 1037–1092

Exact eigenfunctions for the Lax spectrum

By using the bilinear form, we computed eigenfunctions for the Lax spectrum:

$$\Sigma = [\lambda_0, \lambda_0 + k] \cup [\sigma, \infty), \qquad \lambda_0 := -\frac{c+k}{2}.$$

For $[\sigma, \infty)$, $q^- = 0$ and q^+ is analytic in $\mathbb{C}_+$ and bounded as $\mathrm{Im}(x) \to +\infty$:

$$q^+ = e^{i(\lambda-\sigma)x + i(\lambda^2-1)t} \frac{1 + e^{ik(x-ct)+\phi}}{1 + e^{ik(x-ct)-\phi}}.$$

For $[\lambda_0, \lambda_0 + k]$, both q^+ and q^- are bounded as $\mathrm{Im}(x) \to \pm\infty$ and co-periodic:

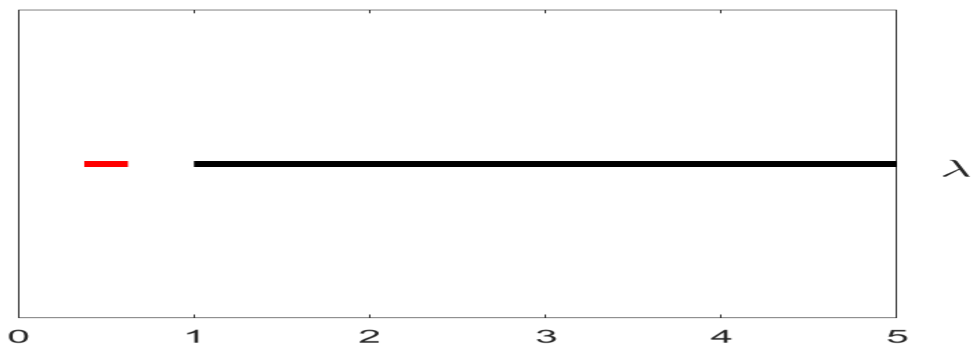
$$q^+ = \frac{1}{1 + e^{ik(x-ct)-\phi}} \left[1 + \frac{c + 2\lambda + k}{c + 2\lambda - k} e^{ik(x-ct)-\phi} \right],$$

$$q^- = \frac{1}{1 + e^{-ik(x-ct)-\phi}} \left[e^{-ik(x-ct)-\phi} + \frac{c + 2\lambda + k}{c + 2\lambda - k} \right].$$

We have $\oint q^- dx = 0$ at $\lambda = \lambda_0$, for which $c + 2\lambda + k = 0$.

P. Gérard, T. Kappeler, Comm. Pure Appl. Math. **74** (2021) 1685–1747

No difference between isolated and embedded bands?



- If $\sigma = +1$, then $[\lambda_0, \lambda_0 + k] \subset (0,1)$ is isolated from $[1, \infty)$.

- If $\sigma = -1$, then either $c \in (k+2, \infty)$ for which $[\lambda_0, \lambda_0 + k]$ is isolated from $[-1, \infty)$ or $c \in (-\infty, -k)$ for which $[\lambda_0, \lambda_0 + k]$ is embedded into $[-1, \infty)$.

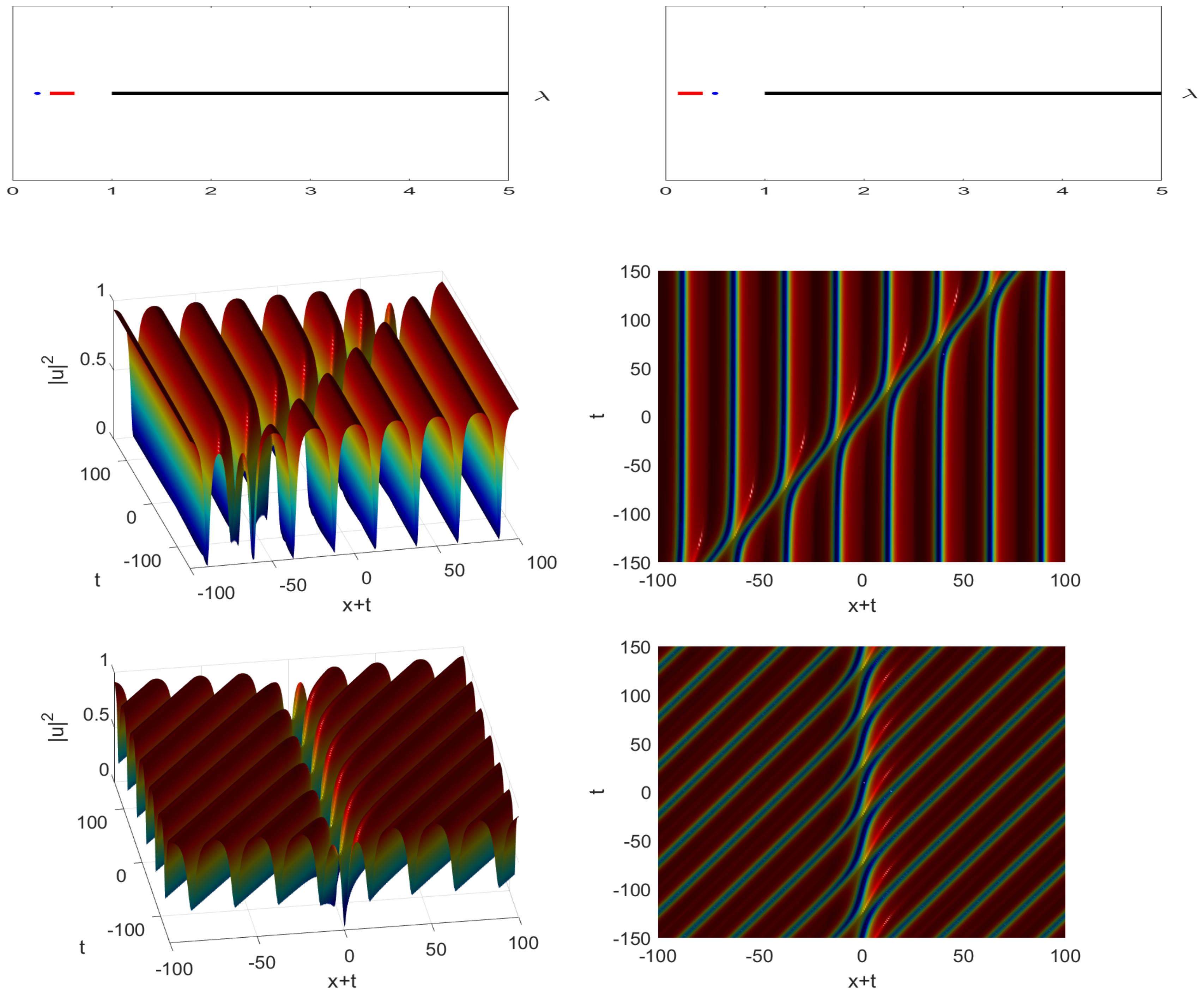
However, all families of traveling periodic waves are symmetric about the midpoint:

$$\sigma = +1: \qquad c + 1 \in (-1 + k, 1 - k), \qquad k \in (0,1)$$
$$\sigma = -1: \qquad c - 1 \in (-\infty, -1 - k) \cup (1 + k, \infty), \qquad k \in (0, \infty).$$

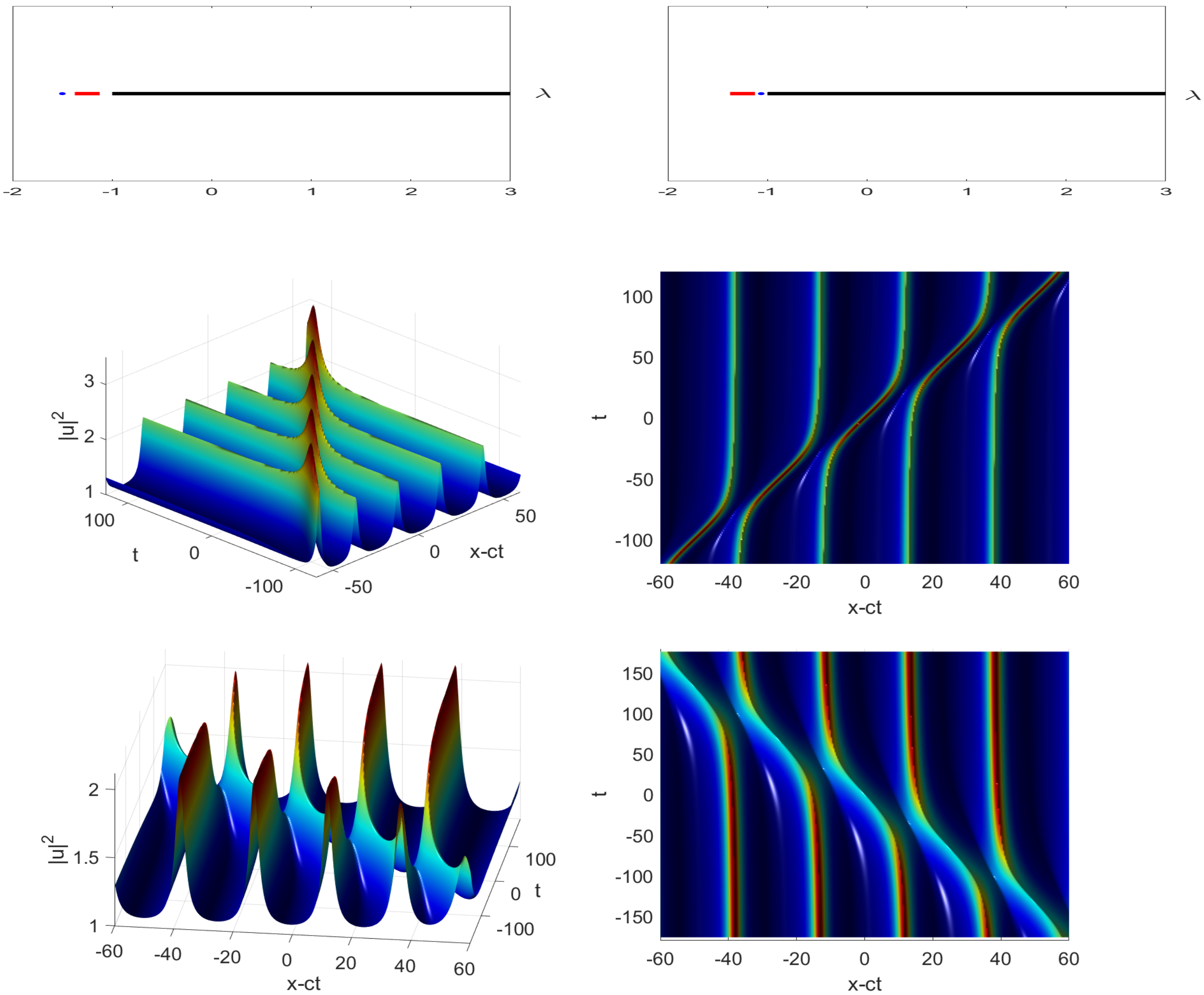
with no difference in dynamics.

In both defocusing and focusing case, we have only obtained breathers steadily propagating on the traveling periodic background. We observed no rogue waves or instability of the traveling periodic wave.

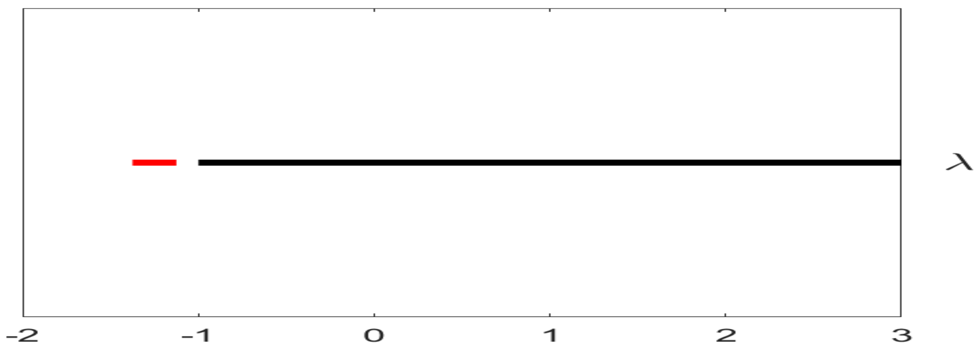
Section 4: Breathers in the defocusing case $\sigma = +1$



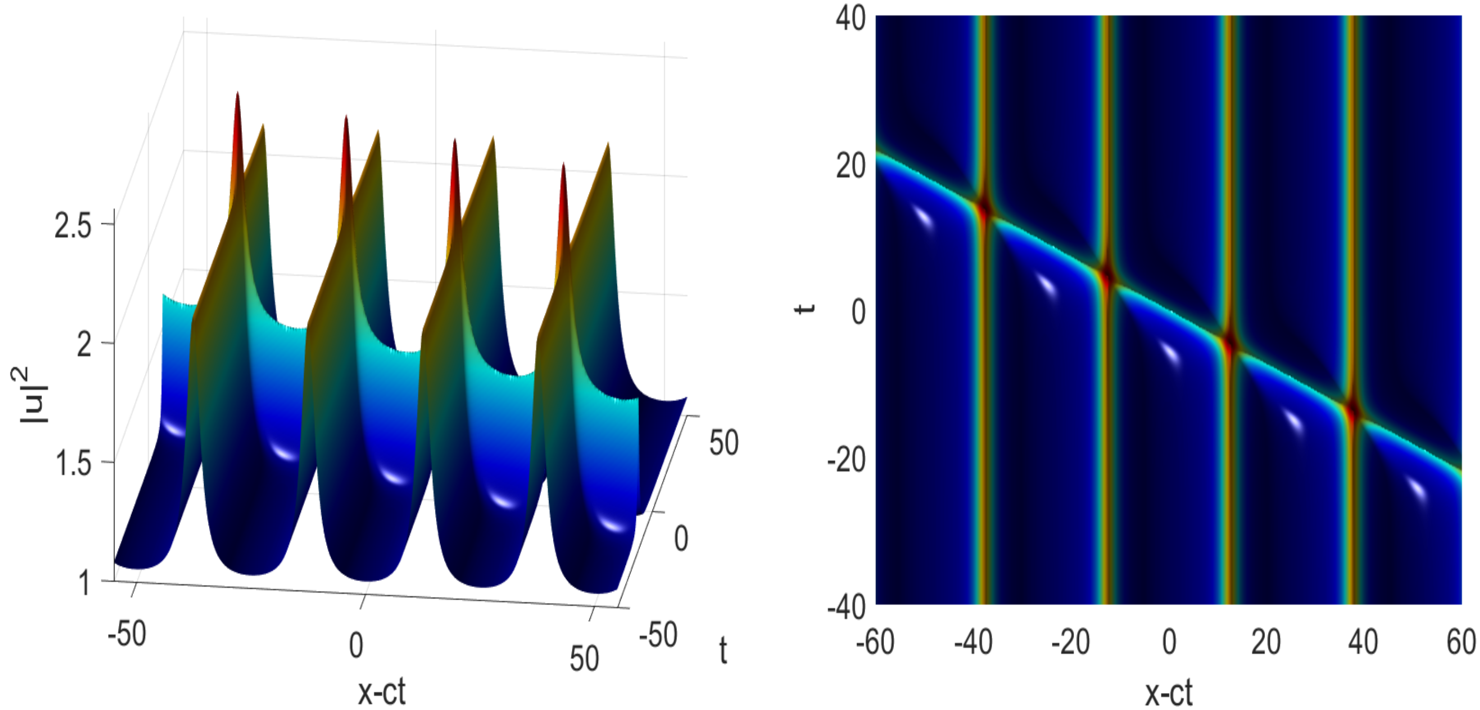
Breathers in the focusing case $\sigma = -1$



Breathers in the focusing case $\sigma = -1$



An embedded eigenvalue $-\frac{c_b}{2} \in [-1, \infty)$ can be added to the Lax spectrum.



Section 5. Conclusion

We have considered stability of the constant background and breathers on the traveling periodic wave background in the NLS–BO equation.

- Defocusing case: Constant background is linearly and nonlinearly stable, dark breathers propagate on the TW background.

- Focusing case: Both bright and dark breathers propagate on the TW background, no rogue waves or instabilities are detected, no difference in dynamics between isolated or embedded eigenvalues.

Work in progress:

- Proving spectral stability of the TW background with a complete set of eigenfunctions satisfying the linearized NLS–BO equation.

- Analysis of the nonlinear stability and possible blowup in the focusing case.

MANY THANKS FOR YOUR ATTENTION