Moving gap solitons in periodic potentials

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Canada

Joint work with Guido Schneider (Institute of Analysis, Modeling and Dynamics, University of Stuttgart, Germany)

References:

Applicable Analysis, **86**, 1017-1036 (2007)

Mathematical Methods in the Applied Sciences, **31**, 1739-1760 (2008)

Motivations

Examples:

Complex-valued Maxwell equation

$$E_{xx} - \left(1 + V(x) + \sigma |E|^2\right) E_{tt} = 0$$

and the Gross–Pitaevskii equation

$$iE_t = -E_{xx} + V(x)E + \sigma |E|^2 E,$$

where $E(x,t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$, $V(x) = V(x + 2\pi)$, and $\sigma = \pm 1$.

Gap solitons are localized stationary solutions of nonlinear PDEs with space-periodic coefficients which reside in a spectral gap of the associated linear Schrödinger operator.

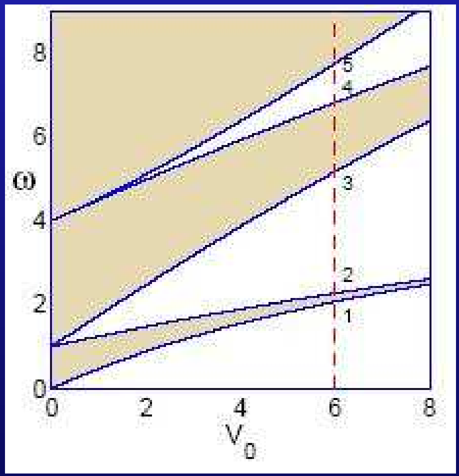
Existence of stationary solutions

Time-periodic solutions $E(x,t) = U(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy the stationary nonlinear equation with a periodic potential

$$\omega U(x) = -U''(x) + V(x)U(x) + \sigma |U|^2 U(x)$$

The associated Schrödinger equation is

$$\begin{cases} -u''(x) + V(x)u(x) = \omega u(x), \\ u(2\pi) = e^{i2\pi k} u(0), \end{cases}$$



Existence results

Previous results:

- Construction of multi-humped gap solitons in Alama-Li (1992)
- Bifurcations of gap solitons from band edges in Kupper-Stuart (1990) and Heinz-Stuart (1992)
- Multiplicity of branches of gap solitons in Heinz (1995)
- Existence of critical points of energy with L^2-normalization in Buffoni-Esteban-Sere (2006)

Theorem:[Stuart, 1995; Pankov, 2005] Let $V(x)$ be a real-valued bounded periodic potential. Let ω be in a finite gap of the spectrum of $L = -\nabla^2 + V(x)$. There exists a non-trivial weak solution $U(x) \in H^1(\mathbb{R})$, which decays exponentially as $|x| \to \infty$.

Illustration of solution branches

D.P., A. Sukhorukov, Yu. Kivshar, PRE **70**, 036618 (2004)

$V(x) = V_0 \sin^2(x)$ with $V_0 = 1$ and $\sigma = -1$:

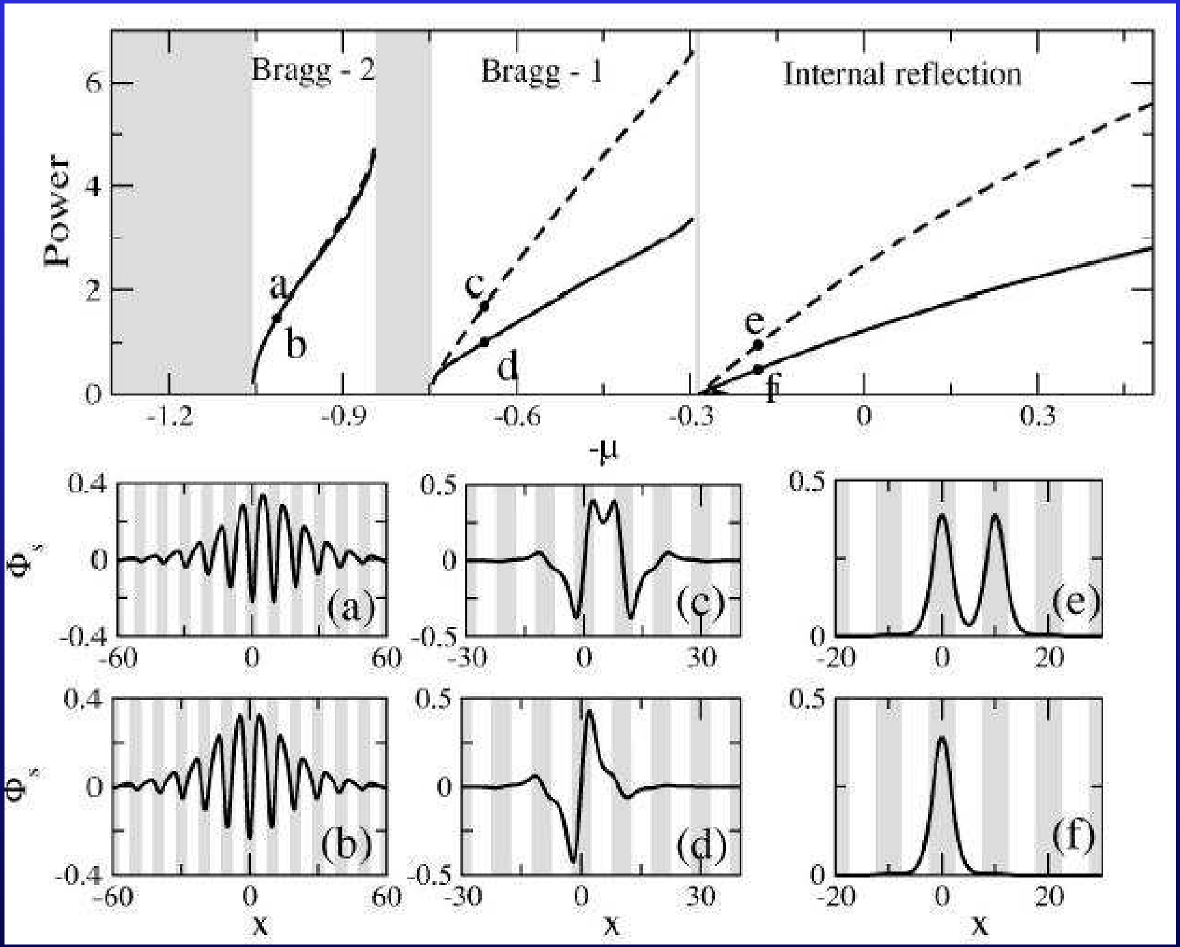
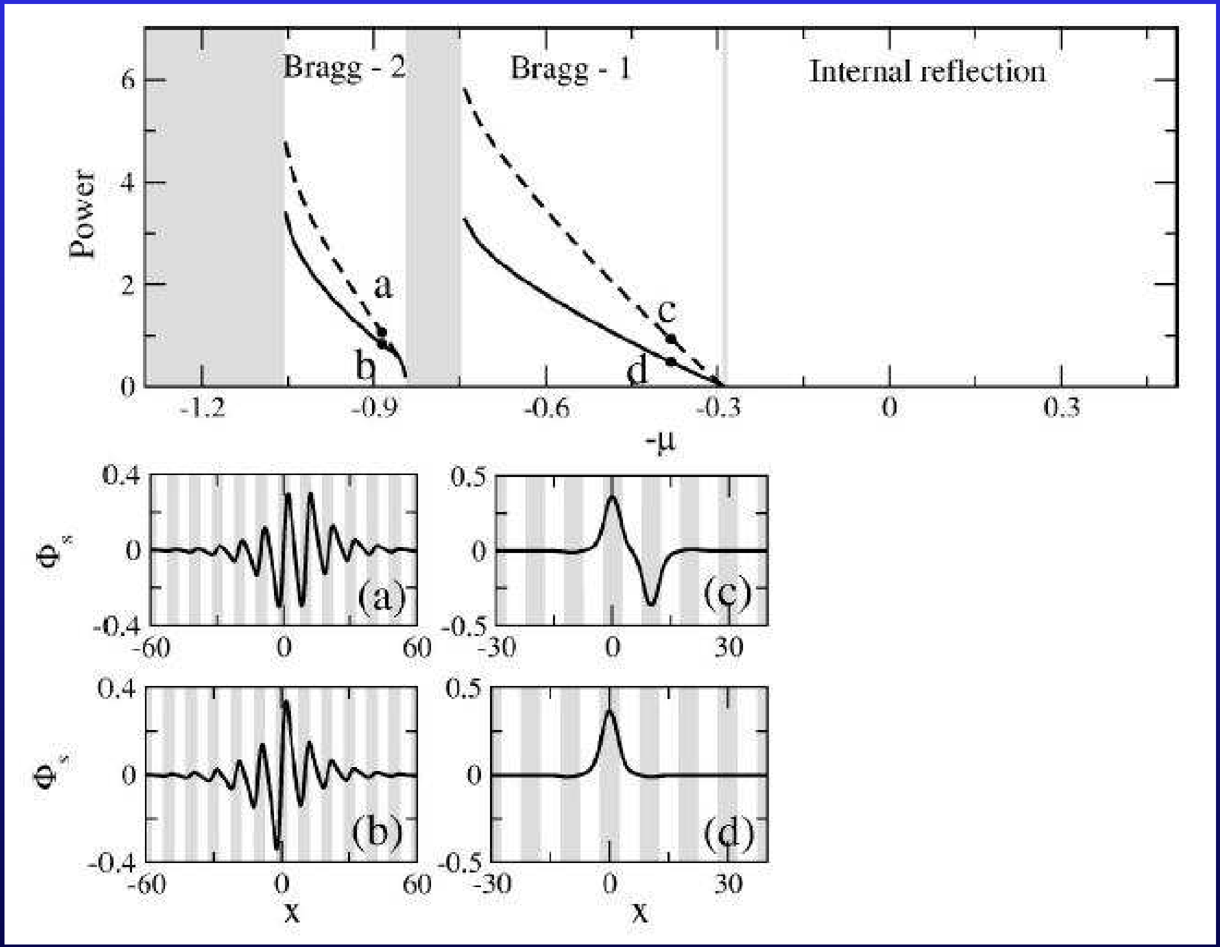


Illustration of solution branches

D.P., A. Sukhorukov, Yu. Kivshar, PRE **70**, 036618 (2004)
$V(x) = V_0 \sin^2(x)$ with $V_0 = 1$ and $\sigma = +1$:



Asymptotic reductions

The nonlinear elliptic problem with a periodic potential can be reduced asymptotically to the following problems:

- Coupled-mode (Dirac) equations for small potentials

$$\begin{cases} i(a_t + a_x) + \alpha b = \sigma(|a|^2 + 2|b|^2)a \\ i(b_t - b_x) + \alpha a = \sigma(2|a|^2 + |b|^2)b \end{cases}$$

- Envelope (NLS) equations for finite potentials near band edges

$$ia_t + a_{xx} + \sigma|a|^2 a = 0$$

- Lattice (dNLS) equations for large or long-period potentials

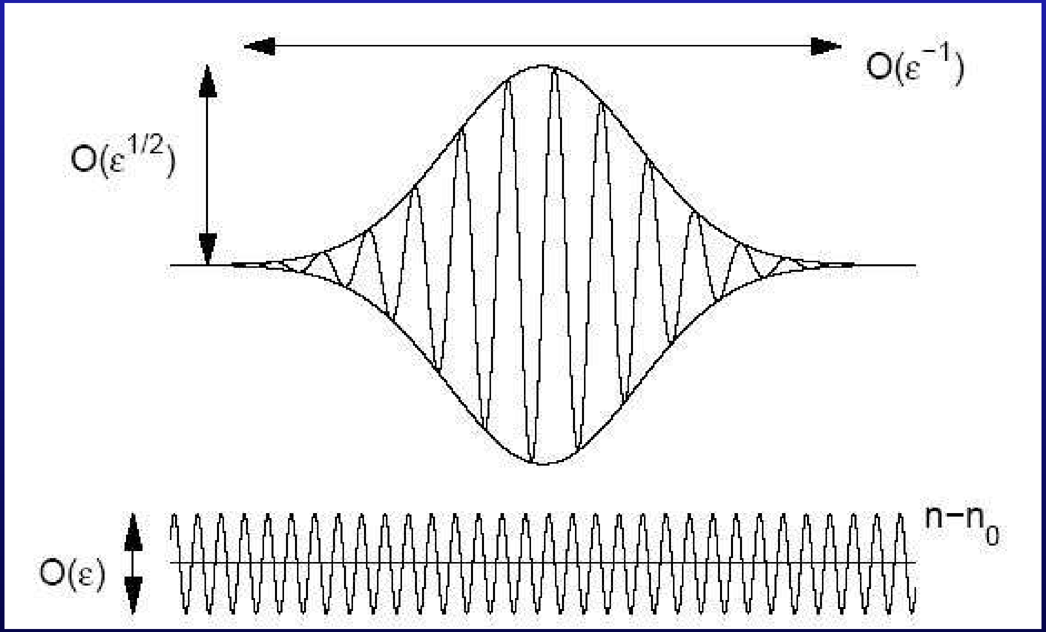
$$i\dot{a}_n + \alpha\left(a_{n+1} + a_{n-1}\right) + \sigma|a_n|^2 a_n = 0.$$

Localized solutions of reduced equations exist in the analytic form.

Formal coupled-mode theory

If $V(x) \equiv 0$, then 2π-periodic or 2π-antiperiodic Bloch functions exist for $\omega = \omega_n = \frac{n^2}{4}$, where $n \in \mathbb{Z}$. Let $\omega = \omega_1$ and consider the asymptotic multi-scale expansion

$$E(x,t) = \sqrt{\epsilon}\left[a(\epsilon x, \epsilon t)e^{\frac{ix}{2}} + b(\epsilon x, \epsilon t)e^{-\frac{ix}{2}} + \mathrm{O}(\epsilon)\right] e^{-\frac{it}{4}}.$$



Coupled-mode equations

The vector $(a, b) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}^2$ satisfies asymptotically the coupled-mode system:

$$\begin{cases} i(a_T + a_X) + V_1 b = \sigma(|a|^2 + 2|b|^2)a, \\ i(b_T - b_X) + V_{-1} a = \sigma(2|a|^2 + |b|^2)b, \end{cases}$$

where $X = \epsilon x$, $T = \epsilon t$, and $V_1 = \bar{V}_{-1}$ are Fourier coefficients of $V(x)$ at $e^{\pm ix}$.

The dispersion relation of the linearized coupled-mode equation is

$$(\omega - \omega_1)^2 = \epsilon^2 |V_1|^2 + k^2.$$

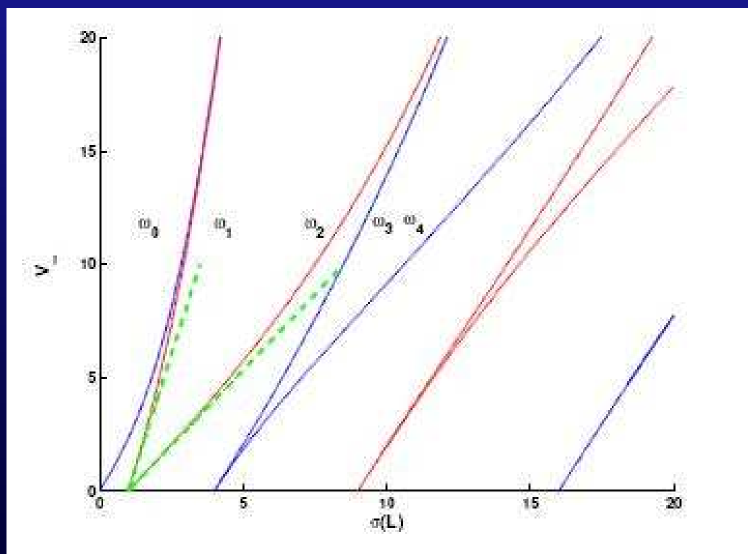
Stationary gap solitons

Stationary gap solitons are obtained in the analytic form

$$a(X,T) = a(X)e^{-i\Omega T}, \quad b(X,T) = b(X)e^{-i\Omega T},$$

where $\kappa = \sqrt{|V_1|^2 - \Omega^2}$ and $|\Omega| < |V_1|$, and

$$a(X) = \bar{b}(X) = \frac{\sqrt{2}}{\sqrt{3}} \frac{\sqrt{|V_1|^2 - \Omega^2}}{\sqrt{|V_1| - \Omega}\cosh(\kappa X) + i\sqrt{|V_1| + \Omega}\sinh(\kappa X)}.$$



Moving gap solitons

Moving gap solitons are obtained in the analytic form

$$a = \left(\frac{1+c}{1-c}\right)^{1/4} A(\xi)e^{-i\mu\tau},\ b = \left(\frac{1-c}{1+c}\right)^{1/4} B(\xi)e^{-i\mu\tau},\ |c| < 1,$$

where

$$\xi = \frac{X - cT}{\sqrt{1-c^2}}, \quad \tau = \frac{T - cX}{\sqrt{1-c^2}}$$

and, since $|A|^2 - |B|^2$ is constant in $\xi \in \mathbb{R}$, then

$$A = \phi(\xi)e^{i\varphi(\xi)}, \qquad B = \bar{\phi}(\xi)e^{i\varphi(\xi)},$$

with ϕ and φ being solutions of the system

$$\varphi' = \frac{-2c\sigma|\phi|^2}{(1-c^2)}, \quad i\phi' = V_1\bar{\phi} - \mu\phi + \sigma\frac{(3-c^2)}{(1-c^2)}|\phi|^2\phi.$$

Questions and Answers

Question 1: Can we justify the use of the coupled-mode theory to approximate stationary gap solitons?

Answer 1: **YES:** we can measure a small approximation error of stationary solutions in $H^1(\mathbb{R})$.

Question 2: Can we justify the use of the coupled-mode theory to approximate moving gap solitons?

Answer 2: **NO:** the small approximation error of traveling solutions is controlled on a large but finite interval and the gap soliton is surrounded by a train of small-amplitude almost-periodic waves.

Time-dependent coupled-mode system

Theorem:[Goodman-Weinstein-Holmes, 2001; Schneider-Uecker, 2001:] Let $(a, b) \in C([0, T_0], H^3(\mathbb{R}, \mathbb{C}^2))$ be solutions of the time-dependent coupled-mode system for a fixed $T_0 > 0$. There exists $\epsilon_0, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the Gross–Pitaevskii equation has a local solution $E(x, t)$ and

$$\|E(x,t) - \sqrt{\epsilon}\left[a(\epsilon x, \epsilon t)e^{i(kx-\omega t)} + b(\epsilon x, \epsilon t)e^{i(-kx-\omega t)}\right]\|_{H^1(\mathbb{R})} \leq C\epsilon$$

for some (k, ω) and any $t \in [0, T_0/\epsilon]$.

Remark: We would like to consider stationary and moving gap solitons in $H^1(\mathbb{R})$ for all $t \in \mathbb{R}$.

Spatial dynamics formulation

Set $E(x,t) = e^{-i\omega t}\psi(x,y)$ with $y = x - ct$ and a parameter ω. For traveling solutions, $c \neq 0$ and we set $c > 0$. Then,

$$\left(\omega - ic\partial_y + \partial_x^2 + 2\partial_x\partial_y + \partial_y^2\right)\psi = \epsilon V(x)\psi + \epsilon\sigma|\psi|^2\psi.$$

We consider functions $\psi(x,y)$ being 2π-periodic or 2π-antiperiodic in x and bounded in y. Therefore,

$$\psi(x,y) = \sum_{m\in\mathbb{Z}'} \psi_m(y) e^{\frac{i}{2}mx},$$

such that $\psi_m(y)$ satisfy the nonlinear system of coupled ODEs:

$$\psi_m'' + i(m-c)\psi_m' + \left(\omega - \frac{m^2}{4}\right)\psi_m = \epsilon \sum_{m_1\in\mathbb{Z}'} V_{m-m_1}\psi_{m_1} + \epsilon \mathrm{N.T.}$$

Eigenvalues of the spatial dynamics

Linearization of the system with $\psi_m(y) = e^{\kappa y}\delta_{m,m_0}$ gives roots $\kappa = \kappa_m$ in the quadratic equation

$$\kappa^2 + i(m-c)\kappa + \omega - \frac{m^2}{4} = 0, \qquad \forall m \in \mathbb{Z}'.$$

- If $\omega = \frac{n^2}{4}$, there is a double zero root $\kappa = 0$ with modes $m = \{n, -n\}$.
- For $m > m_0 = \left[\frac{n^2+c^2}{2c}\right]$, all roots κ are complex-valued.
- For $m \leq m_0$, all roots κ are purely imaginary and semi-simple of maximal multiplicity three.

M. Groves, G. Schneider, Comm. Math. Phys. **219**, 489 (2001)

Assumptions of the main theorem

Assumption: Let $V(x)$ be a smooth 2π-periodic real-valued function with zero mean and symmetry $V(x) = V(-x)$ on $x \in \mathbb{R}$, such that

$$V(x) = \sum_{m \in \mathbb{Z}} V_{2m} e^{imx} : \quad \sum_{m \in \mathbb{Z}} (1+m^2)^s |V_{2m}|^2 < \infty,$$

for some $s \geq 0$, where $V_0 = 0$ and $V_{2m} = V_{-2m} = \bar{V}_{-2m}$.

Definition: The moving gap soliton of the coupled-mode system is said to be a reversible homoclinic orbit if (A, B) decays to zero at infinity and $A(\xi) = \bar{A}(-\xi)$, $B(\xi) = \bar{B}(-\xi)$.

Main theorem for traveling solutions

Theorem: There exists $\epsilon_0, L, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the Gross–Pitaevskii equation has a solution in the form $E(x,t) = e^{-i\omega t}\psi(x,y)$, where $y = x - ct$ and the function $\psi(x,y)$ is a periodic (anti-periodic) function of x for even (odd) n, satisfying the reversibility constraint $\psi(x,y) = \bar{\psi}(x,-y)$, and

$$\left|\psi(x,y) - \epsilon^{1/2}\left(a_\epsilon(\epsilon y)e^{\frac{inx}{2}} + b_\epsilon(\epsilon y)e^{-\frac{inx}{2}}\right)\right| \leq C_0 \epsilon^{N+1/2},$$

for all $x \in \mathbb{R}$ and $y \in [-L/\epsilon^{N+1}, L/\epsilon^{N+1}]$.

Here $a_\epsilon(Y) = a(Y) + \mathrm{O}(\epsilon)$, $Y = \epsilon y$ is an exponentially decaying reversible solution, where $a(Y)$ is a solution of the coupled-mode system with $Y = X - cT$.

Hamiltonian formulation

Let $\phi_m(y) = \psi'_m(y) - \frac{i}{2}(c-m)\psi_m(y)$ and rewrite the system

$$\left\{ \begin{array}{rcl} \frac{d\psi_m}{dy} & = & \phi_m + \frac{i}{2}(c-m)\psi_m \\ \frac{d\phi_m}{dy} & = & -\frac{1}{4}\left(n^2 + c^2 - 2cm\right)\psi_m + \frac{i}{2}(c-m)\phi_m - \epsilon\Omega\psi_m + \text{N.T.} \end{array} \right.$$

The system is Hamiltonian in canonical variables $(\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}})$ on the phase space

$$X = \left\{ (\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}) \in l^2_s(\mathbb{Z}, \mathbb{C}^4) \right\},$$

where $l^2_s(\mathbb{Z})$ is a Banach algebra for any $s > \frac{1}{2}$.

Symmetries

Solutions are invariant under the reversibility transformation

$$\boldsymbol{\psi}(y) \mapsto \bar{\boldsymbol{\psi}}(-y), \quad \boldsymbol{\phi}(y) \mapsto -\bar{\boldsymbol{\phi}}(-y), \quad \forall y \in \mathbb{R}.$$

and the gauge transformation

$$\boldsymbol{\psi}(y) \mapsto e^{i\alpha}\boldsymbol{\psi}(y), \;\; \boldsymbol{\phi}(y) \mapsto e^{i\alpha}\boldsymbol{\phi}(y), \qquad \forall \alpha \in \mathbb{R}.$$

Reversible solutions satisfy the constraints:

$$\boldsymbol{\psi}(-y) = \bar{\boldsymbol{\psi}}(y), \;\; \boldsymbol{\phi}(-y) = -\bar{\boldsymbol{\phi}}(y), \;\; \forall y \in \mathbb{R},$$

which means that the trajectory intersects the reversibility surface

$$\Sigma_r = \left\{ (\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}) \in D : \quad \mathrm{Im}\boldsymbol{\psi} = 0, \;\; \mathrm{Re}\boldsymbol{\phi} = 0 \right\}.$$

Canonical transformations

Let $\mathbb{Z}_- = \{m \in \mathbb{Z}' : \ m \leq m_0\}$, $\mathbb{Z}_+ = \{m \in \mathbb{Z}' : \ m > m_0\}$ and

$$\mathbb{Z}_- : \psi_m = \frac{c_m^+ + c_m^-}{\sqrt[4]{n^2 + c^2 - 2cm}}, \phi_m = \frac{i}{2}\sqrt[4]{n^2 + c^2 - 2cm}(c_m^+ - c_m^-),$$

$$\mathbb{Z}_+ : \psi_m = \frac{c_m^+ + c_m^-}{\sqrt[4]{2cm - n^2 - c^2}}, \phi_m = \frac{1}{2}\sqrt[4]{2cm - n^2 - c^2}(c_m^+ - c_m^-).$$

The new Hamiltonian system is rewritten in new canonical variables

$$\forall m \in \mathbb{Z}_- : \quad \frac{dc_m^+}{dy} = i\frac{\partial H}{\partial \bar{c}_m^+}, \quad \frac{dc_m^-}{dy} = -i\frac{\partial H}{\partial \bar{c}_m^-},$$

$$\forall m \in \mathbb{Z}_+ : \quad \frac{dc_m^+}{dy} = -\frac{\partial H}{\partial \bar{c}_m^-}, \quad \frac{dc_m^-}{dy} = \frac{\partial H}{\partial \bar{c}_m^+},$$

where H is a new Hamiltonian functions in variables $\mathbf{c}^+$ and $\mathbf{c}^-$.

Truncated coupled-mode system

The new Hamiltonian function is

$$H = \sum_{m \in \mathbb{Z}_-} \left(k_m^+ |c_m^+|^2 - k_m^- |c_m^-|^2 \right) + \sum_{m \in \mathbb{Z}_+} \left(\kappa_m^- c_m^- \bar{c}_m^+ - \kappa_m^+ c_m^+ \bar{c}_m^- \right) + \text{N.T.}$$

Consider the subspace

$$S = \{ c_m^+ = 0, \; \forall m \in \mathbb{Z} \backslash \{n\}, \quad c_m^- = 0, \; \forall m \in \mathbb{Z} \backslash \{-n\} \}$$

and truncate H on the subspace S:

$$H|_S = \epsilon \left[\frac{\Omega |c_n^+|^2}{n - c} + \frac{\Omega |c_{-n}^-|^2}{n + c} - \frac{V_{2n} (\bar{c}_n^+ c_{-n}^- + c_n^+ \bar{c}_{-n}^-)}{\sqrt{n^2 - c^2}} + \text{N.T.} \right].$$

The Hamiltonian system for (c_n^+, c_n^-) is nothing but the coupled-mode system for $a = \frac{c_n^+}{\sqrt{n-c}}$ and $b = \frac{c_{-n}^-}{\sqrt{n+c}}$ in $Y = \epsilon y$.

Extended coupled-mode system

Using near-identity canonical transformations, we can obtain the new Hamiltonian function in the form

$$H = \sum_{m\in\mathbb{Z}_-} \left(k_m^+|c_m^+|^2 - k_m^-|c_m^-|^2\right) + \sum_{m\in\mathbb{Z}_+} \left(\kappa_m^- c_m^- \bar{c}_m^+ - \kappa_m^+ c_m^+ \bar{c}_m^-\right)$$

$$+\epsilon H_S(c_n^+, c_{-n}^-) + \epsilon H_T(c_n^+, c_{-n}^-, \mathbf{c}^+, \mathbf{c}^-) + \epsilon^{N+1} H_R(c_n^+, c_{-n}^-, \mathbf{c}^+, \mathbf{c}^-),$$

where H_T is quadratic with respect to $(\mathbf{c}^+, \mathbf{c}^-)$.

If $H_R \equiv 0$, the subspace S is invariant subspace of the Hamiltonian system and dynamics on S is given by

$$\frac{dc_n^+}{dY} = i\frac{\partial H_S}{\partial \bar{c}_n^+}, \qquad \frac{dc_{-n}^-}{dY} = -i\frac{\partial H_S}{\partial \bar{c}_{-n}^+},$$

where $Y = \epsilon y$.

Persistence results

Lemma: There exists a reversible homoclinic orbit of the extended coupled-mode system which satisfies

$$|c_n^+(y)| \leq C_+ e^{-\epsilon\gamma|y|}, \quad |c_{-n}^-(y)| \leq C_- e^{-\epsilon\gamma|y|}, \quad \forall y \in \mathbb{R},$$

for some $\gamma, C_+, C_- > 0$ and sufficiently small ϵ.

Lemma: The linearized system at the zero solution is topologically equivalent for sufficiently small ϵ, except that the double zero eigenvalue at $\epsilon = 0$ split into a pair of complex eigenvalues to the left and right half-planes for $\epsilon > 0$.

Divide the phase space near the zero solution into

$$X = X_h \oplus X_c \oplus X_u \oplus X_s$$

and rewrite the system for $\mathbf{c}_0 + \mathbf{c}_h \in X_h$ and $\mathbf{c} \in X_c \oplus X_u \oplus X_s$.

Local center-stable manifold

Theorem: Let $\mathbf{a} \in X_c$, $\mathbf{b} \in X_s$ and $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ be small:

$$\|\mathbf{a}\|_{X_c} \le C_a \epsilon^N, \quad \|\mathbf{b}\|_{X_s} \le C_b \epsilon^N, \quad |\alpha_1| + |\alpha_2| \le C_\alpha \epsilon^N.$$

There exists a family of local solutions $\mathbf{c}_h = \mathbf{c}_h(y; \mathbf{a}, \mathbf{b}, \alpha_1, \alpha_2)$ and $\mathbf{c} = \mathbf{c}(y; \mathbf{a}, \mathbf{b}, \alpha_1, \alpha_2)$ such that

$$\mathbf{c}_c(0) = \mathbf{a}, \quad \mathbf{c}_s = e^{y\Lambda_s}\mathbf{b} + \tilde{\mathbf{c}}_s(y), \quad \mathbf{c}_h = \alpha_1 \mathbf{s}_1(y) + \alpha_2 \mathbf{s}_2(y) + \tilde{\mathbf{c}}_h(y),$$

where $\tilde{\mathbf{c}}_s(y)$ and $\tilde{\mathbf{c}}_h(y)$ are uniquely defined and the family of local solutions satisfies the bound

$$\sup_{\forall y \in [0, L/\epsilon^{N+1}]} \|\mathbf{c}_h(y)\|_{X_h} \le C_h \epsilon^N, \qquad \sup_{\forall y \in [0, L/\epsilon^{N+1}]} \|\mathbf{c}(y)\|_{X_h^\perp} \le C \epsilon^N,$$

for some constants $C_h, C > 0$.

Proof of the main theorem

The local center-stable manifold is extended to a local solution on $y \in [-y_0, y_0]$ if it intersects the reversibility surface Σ_r.

Since $\mathbf{c}_c(0) = \mathbf{a}$ is arbitrary, we can set immediately

$$\mathrm{Im}(\mathbf{a})^+_m = 0,\ \forall m \in \mathbb{Z}_- \backslash \{n\},\ \ \mathrm{Im}(\mathbf{a})^-_m = 0,\ \forall m \in \mathbb{Z}_- \backslash \{-n\}.$$

The other parameters $\mathbf{b}$ and (α_1, α_2) are not however the initial conditions. They satisfy the set of reversibility constraints

$$\mathrm{Re} b_m + \mathrm{Re}(\tilde{\mathbf{c}}_s)_m(0) = \mathrm{Re}(\mathbf{c}_u)_m(0),\ \mathrm{Im} b_m + \mathrm{Im}(\tilde{\mathbf{c}}_s)_m(0) = -\mathrm{Im}(\mathbf{c}_u)_m(0$$

and

$$\mathrm{Im} c_n^+(0) = 0, \quad \mathrm{Im} c_{-n}^-(0) = 0.$$

The first set is solved by the Implicit Function Theorem. The second set is satisfied if $\alpha_1 = \alpha_2 = 0$, since the kernel does not satisfy the reversibility but the inhomogeneous solution for $\mathbf{c}_h$ does.