# Inverse scattering for the massive Thirring model

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**With Aaron Saalmann (Cologne) and Yusuke Shimabukuro (Taipei)**

Workshop on "Inverse Scattering and Dispersive PDEs in 1D"
Fields Institute, August 8-11, 2017

# Massive Dirac equations in 1D

Massive Dirac equations in one spatial dimension can be written as

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u, v), \end{cases}$$

where $W(u, v) : \mathbb{C}^2 \to \mathbb{R}$ satisfies the following three conditions:

- symmetry $W(u, v) = W(v, u)$;
- gauge invariance $W(e^{i\theta}u, e^{i\theta}v) = W(u, v)$ for any $\theta \in \mathbb{R}$;
- quartic polynomial in $(u, v)$ and $(\bar{u}, \bar{v})$.

Applications include

- Periodic lattices (optics/photonics) - $W = |u|^4 + 4|u|^2|v|^2 + |v|^4$.
- General relativity - $W = (\bar{u}v + u\bar{v})^2$ (Gross–Neveu, Soler, 1974)
- Spinors - $W = |u|^2|v|^2$ (Thirring, 1958)

# Massive Thirring Model (MTM)

$$\left\{ \begin{array}{l} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} i\psi_t - \varphi_x - \psi = (\psi^2 + \varphi^2)\bar{\psi}, \\ i\varphi_t + \psi_x + \varphi = (\psi^2 + \varphi^2)\bar{\varphi}. \end{array} \right.$$

Global solutions exist in $H^1(\mathbb{R})$ [Goodman *et al.* (2003)] or in $L^2(\mathbb{R})$ [Candy (2011), Huh-Moon (2015)].

Three conserved quantities are related to physical symmetries:

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v})\, dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v})\, dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx.$$

Infinitely many conserved quantities exist due to integrability.

# Quick proof of global well-posedness in $H^1(\mathbb{R})$

- $L^2$ conservation gives $\|\mathbf{u}(t)\|_{L^2} = \|\mathbf{u}(0)\|_{L^2}$
- The nonlinear term is canceled in apriori energy estimates:

$$\partial_t \left(|u|^{2p+2} + |v|^{2p+2}\right) + \partial_x \left(|u|^{2p+2} - |v|^{2p+2}\right) = i(p+1)(v\bar{u} - \bar{v}u)(|u|^{2p} - |v|^{2p}).$$

- By Gronwall's inequality, we have

$$\|\mathbf{u}(t)\|_{L^{2p+2}} \leq e^{2|t|}\|\mathbf{u}(0)\|_{L^{2p+2}}, \quad t \in [0, T],$$

which holds for any $p \geq 0$ including $p \to \infty$.

- With the bound on $\|\mathbf{u}(t)\|_{L^\infty}$, one can obtain

$$\frac{d}{dt}\|\partial_x \mathbf{u}(t)\|^2_{L^2} \leq Ce^{4|t|}\|\partial_x \mathbf{u}(t)\|^2_{L^2},$$

hence $\|\partial_x \mathbf{u}(t)\|^2_{L^2}$ does not blow up in a finite time.

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# Existence of solitary waves

Time-periodic space-localized solutions

$$u(x,t) = U_\omega(x) e^{-i\omega t}, \quad v(x,t) = V_\omega(x) e^{-i\omega t}$$

satisfy a system of stationary Dirac equations. They are known in the closed analytic form

$$\begin{cases} u(x,t) = i \sin(\gamma) \operatorname{sech}\left[x \sin\gamma - i\frac{\gamma}{2}\right] \; e^{-it\cos\gamma}, \\ v(x,t) = -i \sin(\gamma) \operatorname{sech}\left[x \sin\gamma + i\frac{\gamma}{2}\right] \; e^{-it\cos\gamma}. \end{cases}$$

- Translations in $x$ and $t$ can be added as free parameters.
- Constraint $\omega = \cos\gamma \in (-1,1)$ exists because of the gap in the linear spectrum $(-\infty,-1] \cup [1,\infty)$.
- Moving solitons can be obtained from the stationary solitons with the Lorentz transformation.

# Orbital stability of solitary waves

## Definition

We say that the solitary wave $e^{-i\omega t}\mathbf{U}_\omega(x)$ is orbitally stable in $X$ if for any $\epsilon > 0$ there is a $\delta > 0$, such that if $\|\mathbf{u}(\cdot,0) - \mathbf{U}_\omega(\cdot)\|_X \leq \delta$ then

$$\inf_{\theta,a\in\mathbb{R}} \|\mathbf{u}(\cdot,t) - e^{-i\theta}\mathbf{U}_\omega(\cdot + a)\|_X \leq \epsilon,$$

for all $t > 0$. Here $X = H^1(\mathbb{R})$ or $X = L^2(\mathbb{R})$.

- Stability of Dirac solitons was mainly studied numerically, e.g., by I. Barashenkov (1998), G. Gottwald (2005), M. Chugunova (2006), A. Comech (2012), A. Saxena (2014), P. Kevrekidis (2014), ...
- For Soler and Thirring models, Dirac solitons were found to be spectrally stable. For other models (e.g. for optical lattices), Dirac solitons are unstable for some parameter values.

# Outline of presentation

1. Stability argument for MTM based on energy functionals
2. Stability argument for MTM based on Backlund transformation
3. Scattering results for MTM based on transformations to the Zakharov–Shabat spectral problem

Some old references:


- E.A. Kuznetsov and A.V. Mikhailov (1977) - "On the complete integrability of the 2D classical MTM"
- D.Kaup and A.C. Newell (1977) - "On the Coleman correspondence and the solution of MTM"
- J. Villarroel (1991) - "The DBAR problem and the MTM"

# 1. Stability argument based on energy functionals

Three conserved quantities are related to physical symmetries:

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v}\right) dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v}\right) dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx,$$

Since the quadratic part of $H$ is sign-indefinite, Dirac soliton can not be a constrained minimizer of $H$.

Another conserved quantity $R$ exists in $H^1(\mathbb{R})$:

$$R = \int_{\mathbb{R}} \Big[ |u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x\overline{u} - \overline{u}_x u)(|u|^2 + 2|v|^2) + \dots$$

$$-(u\overline{v} + \overline{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \Big] dx.$$

# The energy functionals

- Critical points of $H + \omega Q$ for a fixed $\omega \in (-1, 1)$ satisfy the stationary MTM equations. After the reduction $(u, v) = (U, \overline{U})$, we obtain the first-order equation

$$i \frac{dU}{dx} - \omega U + \overline{U} = 2|U|^2 U.$$

The MTM soliton $U = U_\omega$ satisfies the first-order equation.

- Critical points of $R + \Omega Q$ for some fixed $\Omega \in \mathbb{R}$ satisfy another system of differential equations. After the reduction $(u, v) = (U, \overline{U})$, we obtain the second-order equation

$$\frac{d^2 U}{dx^2} + 6i|U|^2 \frac{dU}{dx} - 6|U|^4 U + 3|U|^2 \bar{U} + U^3 = \Omega U.$$

$U = U_\omega$ also satisfies the second-order equation if $\Omega = 1 - \omega^2$.

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# Convexity of the energy functional

**Theorem (P–Shimabukuro, 2014)**

*There is $\omega_0 \in (0,1]$ such that for any fixed $\omega = \cos\gamma \in (-\omega_0, \omega_0)$, MTM soliton is a local non-degenerate minimizer of $R$ in $H^1(\mathbb{R})$ under the fixed values of $Q$ and $P$.*

Consider the conserved energy functional in $H^1(\mathbb{R})$ by

$$\Lambda_\omega := R + (1-\omega^2)Q, \quad \omega \in (-1,1),$$

where $Q = \|u\|_{L^2}^2 + \|v\|_{L^2}^2$.

- $U_\omega$ is a critical point of $\Lambda_\omega$.
- The second variation of $\Lambda_\omega$ can be block-diagonalized

$$S^T \Lambda''_\omega(U_\omega) S = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

where $L_+$ and $L_-$ are $2 \times 2$ matrix Schrödinger operators.

# Convexity of the energy functional

- For $\omega \in (-\omega_0, \omega_0)$, $\Lambda''_\omega(U_\omega)$ has a simple negative eigenvalue and a double zero eigenvalue for $\omega > 0$ and $\omega < 0$. The zero eigenvalue is quadruple for $\omega = 0$.
- Two constraints are added to fix the values of $Q$ and $P$.
- Two symmetries are included to eliminate translation and rotation.
- The Hessian operator $\Lambda''_\omega(U_\omega)$ is strictly positive under the four constraints. The conserved energy $\Lambda_\omega$ is convex at $U_\omega$ in the constrained $H^1(\mathbb{R})$ space.
- The four constraints can be realized by the choice of four modulation parameters in the soliton orbit:

$$\begin{cases} u(x,t) = i\sin(\gamma)\,\text{sech}\left[x\sin(\gamma) - i\frac{\gamma}{2} - \alpha\right]\ e^{-it\cos(\gamma) - i\beta}, \\ v(x,t) = -i\sin(\gamma)\,\text{sech}\left[x\sin(\gamma) + i\frac{\gamma}{2} - \alpha\right]\ e^{-it\cos(\gamma) - i\beta}, \end{cases}$$

with parameters $\alpha$, $\beta$, frequency $\omega := \cos\gamma$, and speed $c$.

# Orbital stability result

- Strict positivity (coercivity) of the second variation implies

$$\Lambda_\omega(\mathbf{U}_\omega + \mathbf{U}) - \Lambda_\omega(\mathbf{U}_\omega) \geq C\|\mathbf{U}\|_{H^1}^2 + \mathcal{O}(\|\mathbf{U}\|_{H^1}^3),$$

  for perturbation $\mathbf{U} \in H^1(\mathbb{R}; \mathbb{C}^2)$ in the constrained space.

- $R$, $Q$, and $P$ are constant in time $t$ and so is $\Lambda_\omega$.

- A global lower bound is obtained for the solution $\mathbf{u}(t)$ near a modulated orbit of the MTM soliton $\mathbf{U}_\omega$ for every $t \in \mathbb{R}$:

$$\Lambda_\omega(\mathbf{u}) - \Lambda_\omega(\mathbf{U}_\omega) \geq \inf_{\theta, x_0} \|\mathbf{u}(\cdot, t) - e^{i\theta}\mathbf{U}_\omega(\cdot + x_0)\|_{H^1}^2.$$

- This yields orbital stability of MTM solitons in $H^1(\mathbb{R})$ for $\omega \in (-\omega_0, \omega_0)$.

# Recent results in this direction

- Stability of $N$ solitary waves:
  - KdV (Sachs–Maddocks, 1993);
  - NLS (Kapitula, 2006);
  - derivative NLS (Le Coz–Wu, 2017);
- Stability of time-periodic localized breathers:
  - mKdV (Alejo–Munoz, 2013);
  - sine–Gordon (Alejo–Munoz, 2016);
  - Gardner (Alejo, 2017);
- Stability of space-periodic solutions with respect to subharmonic perturbations:
  - KdV (Deconinck–Kaputula, 2010);
  - NLS (Gallay-P, 2014);
  - KP (Haragus–Li–P, 2017);

# 2. Stability argument based on Bäcklund transformation

The Bäcklund transformation $\mathcal{B}$ is a map that takes $(u, v)$ of the MTM to $(\tilde{u}, \tilde{v})$ of the MTM,

$$\mathcal{B} : (u, v) \mapsto (\tilde{u}, \tilde{v}),$$

In particular, the Bäcklund transformation relates zero $\leftrightarrow$ one soliton:

$$(0, 0) \stackrel{\mathcal{B}}{\longleftrightarrow} (u_\lambda, v_\lambda)$$

**Heuristic stability argument by Bäcklund transformation**

$\mathcal{B}$ : stable small solution $\longleftrightarrow$ solution around stable one soliton.

–Merle-Vega-2003 (KdV solitons)
–Mizumachi-P-2012 (NLS solitons)

# Lax operators for the MTM

The MTM is obtained from the compatibility condition of the linear system

$$\vec{\phi}_x = L\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A\vec{\phi},$$

where

$$L = \frac{i}{2}(|v|^2 - |u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}} \begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda} \begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\frac{1}{\lambda^2} - \lambda^2\right)\sigma_3$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2} \begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda} \begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3$$

Kaup–Newell (1977); Kuznetsov–Mikhailov (1977).

# Bäcklund transformation for the MTM

- Let $(u, v)$ be a $C^1$ solution of the MTM system.
- Let $\vec{\phi} = (\phi_1, \phi_2)^t$ be a $C^2$ nonzero solution of the linear system associated with $(u, v)$ and $\lambda = \delta e^{i\gamma/2}$.

A new $C^1$ solution of the MTM system is given by

$$\mathbf{u} = -u \frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} + \frac{2i\delta^{-1} \sin\gamma \bar{\phi}_1 \phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}$$

$$\mathbf{v} = -v \frac{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} - \frac{2i\delta \sin\gamma \bar{\phi}_1 \phi_2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2},$$

A new $C^2$ nonzero solution $\vec{\psi} = (\psi_1, \psi_2)^t$ of the linear system associated with $(\mathbf{u}, \mathbf{v})$ and same $\lambda$ is given by

$$\psi_1 = \frac{\bar{\phi}_2}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}, \quad \psi_2 = \frac{\bar{\phi}_1}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}.$$

# Bäcklund transformation $0 \leftrightarrow 1$ soliton

Let $(u, v) = (0, 0)$ and define

$$\begin{cases} \phi_1 = e^{\frac{i}{4}(\lambda^2-\lambda^{-2})x+\frac{i}{4}(\lambda^2+\lambda^{-2})t}, \\ \phi_2 = e^{-\frac{i}{4}(\lambda^2-\lambda^{-2})x-\frac{i}{4}(\lambda^2+\lambda^{-2})t}. \end{cases}$$

Then, $(\mathbf{u}, \mathbf{v}) = (u_\lambda, v_\lambda)$ is the MTM soliton. If $\lambda = e^{i\gamma/2}$ (stationary case), the vector $\vec{\psi}$ is given by

$$\begin{cases} \psi_1 = e^{\frac{1}{2}x\sin\gamma+\frac{i}{2}t\cos\gamma} \left|\operatorname{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|, \\ \psi_2 = e^{-\frac{1}{2}x\sin\gamma-\frac{i}{2}t\cos\gamma} \left|\operatorname{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|. \end{cases}$$

It decays exponentially as $|x| \to \infty$.

In the opposite direction, if $(u, v) = (u_\lambda, v_\lambda)$ and $\vec{\phi} = \vec{\psi}$, then $(\mathbf{u}, \mathbf{v}) = (0, 0)$.

# Orbital stability result

**Theorem (Contreras–P–Shimabukuro, 2016)**

*Let* $\mathbf{u}(t) \in C(\mathbb{R}; L^2(\mathbb{R}))$ *be a solution of the MTM system and* $\lambda_0 \in \mathbb{C}_I$. *There exist a real positive constant* $\epsilon$ *such that if the initial value* $\mathbf{u}_0 \in L^2(\mathbb{R})$ *satisfies*

$$\|\mathbf{u} - \mathbf{u}_{\lambda_0}(0, \cdot)\|_{L^2} \leq \epsilon,$$

*then for every* $t \in \mathbb{R}$, *there exists* $\lambda \in \mathbb{C}$ *such that* $|\lambda - \lambda_0| \leq C\epsilon$,

$$\inf_{a,\theta \in \mathbb{R}} \|\mathbf{u}(t, \cdot + a) - e^{-i\theta}\mathbf{u}_\lambda(t, \cdot)\|_{L^2} \leq C\epsilon,$$

*where the constant* $C$ *is independent of* $\epsilon$ *and* $t$.

The proof does not require the inverse scattering formalism.

# How does the argument go?

Fix $\lambda_0 \in \mathbb{C}_I$ for a MTM soliton $\mathbf{u}_{\lambda_0}$. Take initial data $\mathbf{u}_0 \in H^2(\mathbb{R})$ s.t. $\|\mathbf{u}_0 - \mathbf{u}_{\lambda_0}\|_{L^2} < \epsilon$ for $\epsilon > 0$ sufficiently small.

1. From a perturbed one-soliton to a small solution at $t = 0$:

   There exists $\lambda \in \mathbb{C}$ and the corresponding $L^2$-solution $\vec{\psi}$ of $\partial_x \vec{\psi} = L(\mathbf{u}_0; \lambda)\vec{\psi}$ such that $|\lambda - \lambda_0| \lesssim \epsilon$. Then, Bäcklund transformation

   $$\mathcal{B}(\vec{\psi}, \lambda) : \mathbf{u}_0 \mapsto \widetilde{\mathbf{u}}_0$$

   yields the estimate

   $$\|\widetilde{\mathbf{u}}_0\|_{L^2} \lesssim \|\mathbf{u}_0 - \mathbf{u}_{\lambda_0}(0, \cdot)\|_{L^2}.$$

2. Time evolution of the small solution in $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$.

   $$\|\widetilde{\mathbf{u}}(t, \cdot)\|_{L^2} = \|\widetilde{\mathbf{u}}_0\|_{L^2}, \quad t \in \mathbb{R}.$$

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$$\|\widetilde{\mathbf{u}}(t, \cdot)\|_{L^2} = \|\widetilde{\mathbf{u}}_0\|_{L^2}, \quad t \in \mathbb{R}.$$

3 From the small solution to the perturbed one-soliton:

For every $t \in \mathbb{R}$, we construct solutions of

$$\vec{\phi}_x = L(\widetilde{\mathbf{u}}(t,\cdot),\lambda)\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A(\widetilde{\mathbf{u}}(t,\cdot),\lambda)\vec{\phi},$$

which is defined with two arbitrary parameters $a(t)$ and $\theta(t)$.

The Bäcklund transformation

$$\mathcal{B}(\vec{\phi},\lambda) : \widetilde{\mathbf{u}}(t,\cdot) \mapsto \mathbf{u}(t,\cdot)$$

yields the estimate

$$\inf_{a,\theta\in\mathbb{R}} \|\mathbf{u}(t,\cdot) - e^{-i\theta}\mathbf{u}_\lambda(t,\cdot + a)\|_{L^2} \lesssim \|\widetilde{\mathbf{u}}(t,\cdot)\|_{L^2} \quad \forall t \in \mathbb{R}.$$

4 Approximating sequence $\mathbf{u}_{0,n}$ in $H^2(\mathbb{R})$ that converges to $\mathbf{u}_0 \in L^2(\mathbb{R})$.

Sequences in $H^2(\mathbb{R})$ produce classical solutions of the MTM, which are compatible with the Lax linear system for $\vec{\phi} \in C^2(\mathbb{R}\times\mathbb{R})$.

3 From the small solution to the perturbed one-soliton:

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yields the estimate

$$\inf_{a,\theta\in\mathbb{R}} \|\mathbf{u}(t,\cdot) - e^{-i\theta}\mathbf{u}_\lambda(t,\cdot+a)\|_{L^2} \lesssim \|\widetilde{\mathbf{u}}(t,\cdot)\|_{L^2} \quad \forall t \in \mathbb{R}.$$

4 Approximating sequence $\mathbf{u}_{0,n}$ in $H^2(\mathbb{R})$ that converges to $\mathbf{u}_0 \in L^2(\mathbb{R})$.

Sequences in $H^2(\mathbb{R})$ produce classical solutions of the MTM, which are compatible with the Lax linear system for $\vec{\phi} \in C^2(\mathbb{R}\times\mathbb{R})$.

# Recent results in this direction

- Asymptotic stability for NLS:
  - 1-soliton (Cuccagna–P, 2014);
  - $N$-solitons with Backlund transformation (Contreras–P, 2014);
  - $N$-solitons with inverse scattering (Saalmann, 2017);

- Global existence for derivative NLS:
  - $N$-solitons with Bäcklund transformation (Shimabukuro–Saalmann–P, 2017);
  - $N$-solitons with inverse scattering (Jenkins–Liu–Perry–Sulem, 2017).

# 3. Scattering results for MTM

Massive Thirring Model (MTM) in physical coordinates

$$\left\{ \begin{array}{l} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{array} \right.$$

is rewritten in characteristic coordinates as

$$\left\{ \begin{array}{l} iu_\tau + v = 2|v|^2 u, \\ -iv_\xi + u = 2|u|^2 v, \end{array} \right.$$

with $\xi = (x-t)/2$ and $\tau = (x+t)/2$.

MTM in characteristic coordinates is related to the linear system

$$\vec{\phi}_\xi = L\vec{\phi} \quad \text{and} \quad \vec{\phi}_\tau = A\vec{\phi},$$

associated with the Kaup–Newell spectral problem

$$L = -i\lambda^2 \sigma_3 + \lambda \begin{pmatrix} 0 & w \\ -\bar{w} & 0 \end{pmatrix}, \quad w = 2ue^{2i\int_\xi^\infty |u|^2 d\xi}.$$

# Direct scattering problem

Kaup-Newel spectral problem

$$\partial_x \psi = -i\lambda^2 \sigma_3 \psi + \lambda Q(u)\psi, \quad Q(u) = \begin{bmatrix} 0 & u \\ -\bar{u} & 0 \end{bmatrix}$$

Jost functions are defined by the asymptotical limits:

$$\Psi_\pm(x;\lambda) \to e^{-i\lambda^2 x \sigma_3} \quad \text{as } x \to \pm\infty.$$

Jost functions in $\Psi_\pm := e^{-i\lambda^2 x\sigma_3}[\varphi_\pm, \phi_\pm]$ satisfy Volterra's equations

$$\varphi_\pm(x;\lambda) = e_1 + \lambda \int_{\pm\infty}^{x} \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\lambda^2(x-y)} \end{bmatrix} Q(u(y))\varphi_\pm(y;\lambda)dy,$$

$$\phi_\pm(x;\lambda) = e_2 + \lambda \int_{\pm\infty}^{x} \begin{bmatrix} e^{-2i\lambda^2(x-y)} & 0 \\ 0 & 1 \end{bmatrix} Q(u(y))\phi_\pm(y;\lambda)dy.$$

Fixed point arguments are not uniform in $\lambda$ as $|\lambda| \to \infty$ if $Q(u) \in L^1(\mathbb{R})$.

# The way around this obstacle

Introduce transformations $m_\pm := T_1\varphi_\pm$ and $n_\pm := T_2\phi_\pm$, where

$$T_1(x;\lambda) = \begin{bmatrix} 1 & 0 \\ -\overline{u}(x) & 2i\lambda \end{bmatrix}, \quad T_2(x;\lambda) = \begin{bmatrix} 2i\lambda & -u(x) \\ 0 & 1 \end{bmatrix},$$

Volterra's integral equations are rewritten for $m_\pm$ and $n_\pm$ as follows:

$$m_\pm(x;z) = e_1 + \int_{\pm\infty}^{x} \begin{bmatrix} 1 & 0 \\ 0 & e^{2iz(x-y)} \end{bmatrix} Q_1(u(y)) m_\pm(y;z)dy,$$

$$n_\pm(x;z) = e_2 + \int_{\pm\infty}^{x} \begin{bmatrix} e^{-2iz(x-y)} & 0 \\ 0 & 1 \end{bmatrix} Q_2(u(y)) n_\pm(y;z)dy,$$

where $z := \lambda^2$ and

$$Q_1(u) = \frac{1}{2i} \begin{bmatrix} |u|^2 & u \\ -2i\overline{u}_x - \overline{u}|u|^2 & -|u|^2 \end{bmatrix}, \quad Q_2(u) = \frac{1}{2i} \begin{bmatrix} |u|^2 & -2iu_x + u|u|^2 \\ -\overline{u} & -|u|^2 \end{bmatrix}.$$

Instead of one Kaup-Newell problem, two Zakharov-Shabat problems!

# A bit of history

Kaup-Newel spectral problem

$$\partial_x \psi = -i\lambda^2 \sigma_3 \psi + \lambda Q(u) \psi, \quad Q(u) = \begin{bmatrix} 0 & u \\ -\overline{u} & 0 \end{bmatrix}$$

is reduced to the two Zakharov–Shabat spectral problems

$$\partial_x \tilde{\psi}_{1,2} = -iz\sigma_3 \tilde{\psi}_{1,2} + Q_{1,2}(u) \tilde{\psi}_{1,2},$$

with $z := \lambda^2$ and

$$Q_1(u) = \frac{1}{2i} \begin{bmatrix} |u|^2 & u \\ -2i\overline{u}_x - \overline{u}|u|^2 & -|u|^2 \end{bmatrix}, \quad Q_2(u) = \frac{1}{2i} \begin{bmatrix} |u|^2 & -2iu_x + u|u|^2 \\ -\overline{u} & -|u|^2 \end{bmatrix},$$

- Threshold on $u$ for the nonexistence of isolated eigenvalues (P, 2011)
- Inverse scattering for derivative NLS (P–Shimabukuro, 2015)
- $Q_1$ appears already in (Kaup–Newell, 1976).

# Choice of spaces

From the condition $Q_{1,2}(u) \in L^1(\mathbb{R})$, where

$$Q_1(u) = \frac{1}{2i}\begin{bmatrix} |u|^2 & u \\ -2i\bar{u}_x - \bar{u}|u|^2 & -|u|^2 \end{bmatrix}, \quad Q_2(u) = \frac{1}{2i}\begin{bmatrix} |u|^2 & -2iu_x + u|u|^2 \\ -\bar{u} & -|u|^2 \end{bmatrix},$$

we realize that $u \in L^1(\mathbb{R}) \cap L^3(\mathbb{R})$ and $\partial_x u \in L^1(\mathbb{R})$ is a natural choice. With $u \in L^\infty(\mathbb{R})$, the transformation matrices $T_{1,2}$ are defined in $L^\infty(\mathbb{R})$.

- There exist unique $L^\infty$ solutions $m_\pm(\cdot; z)$ for every $z \in \mathbb{R}$.
- For every $x \in \mathbb{R}$, $m_\mp(x; \cdot)$, $n_\pm(x; \cdot)$ are continued analytically in $\mathbb{C}^\pm$.
- Limits of $m_\mp(x; z)$, $n_\pm(x; z)$ as $|z| \to \infty$ are defined in $\mathbb{C}^\pm$.

To use Fourier theory, it is better to work in $H^{1,1}(\mathbb{R})$ with $u, \partial_x u \in L^{2,1}(\mathbb{R})$.

The inverse scattering transform for the derivative NLS starts from here with extra constraint $u \in H^2(\mathbb{R})$ due to the time evolution ...

# Global existence for derivative NLS

Recall the Cauchy problem related to the derivative NLS equation

$$\begin{cases} iu_t + u_{xx} + i(|u|^2 u)_x = 0, & t > 0, \\ u|_{t=0} = u_0 \in X = H^s(\mathbb{R}), \end{cases}$$

and the Kaup-Newel spectral problem:

$$(\mathrm{KN}) \quad \partial_x \psi = \left[ -i\lambda^2 \sigma_3 + \lambda Q(u) \right] \psi, \quad \psi \in \mathbb{C}^2.$$

**Theorem (P–Shimabukuro, 2015)**

For every $u_0 \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ such that (KN) admits no eigenvalues or resonances, there exists a unique global solution $u(t,\cdot) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ of the Cauchy problem for every $t \in \mathbb{R}$. Furthermore, the map $u_0 \mapsto u$ is Lipschitz.

The function spaces and the inverse scattering method is different from the one used by (Liu–Perry–Sulem, 2015).

# Back to MTM

Massive Thirring Model (MTM) in physical coordinates

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases}$$

is related to the linear system

$$\vec{\phi}_\xi = L\vec{\phi} \quad \text{and} \quad \vec{\phi}_\tau = A\vec{\phi},$$

associated with the linear operators

$$L = \frac{i}{2}(|v|^2 - |u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\frac{1}{\lambda^2} - \lambda^2\right)\sigma_3$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3$$

# Direct scattering: $|\lambda| > 1$

Spectral problem $\partial_x \psi = L\psi$ can be transformed with $\tilde{\psi} = T(x;\lambda)\psi$ with

$$T(x;\lambda) = \begin{bmatrix} 1 & 0 \\ v(x) & \lambda \end{bmatrix}$$

to the equivalent form

$$\partial_x \tilde{\psi} = \tilde{Q}_1(u,v)\tilde{\psi} + \frac{1}{z}\tilde{Q}_2(u,v)\tilde{\psi} + \frac{i}{4}\left(z - \frac{1}{z}\right)\sigma_3\tilde{\psi},$$

where $z := \lambda^2$ and

$$\tilde{Q}_1 = \begin{bmatrix} \frac{i}{4}(|u|^2 + |v|^2) & -\frac{i}{2}\bar{v} \\ v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u & -\frac{i}{4}(|u|^2 + |v|^2) \end{bmatrix}, \quad \tilde{Q}_2 = \frac{1}{2i}\begin{bmatrix} \bar{u}v & -\bar{u} \\ v + \bar{u}v^2 & -\bar{u}v \end{bmatrix}.$$

This system is useful as $z \to \infty$.

# Direct scattering: $|\lambda| < 1$

Spectral problem $\partial_x \psi = L\psi$ can be transformed with $\hat{\psi} = T(x;\lambda)\psi$ with

$$T(x;\lambda) = \begin{bmatrix} 1 & 0 \\ u(x) & \lambda^{-1} \end{bmatrix}$$

to the equivalent form

$$\partial_x \hat{\psi} = \hat{Q}_1(u,v)\hat{\psi} + z\hat{Q}_2(u,v)\hat{\psi} + \frac{i}{4}\left(z - \frac{1}{z}\right)\sigma_3\hat{\psi},$$

where $z := \lambda^2$ and

$$\hat{Q}_1 = \begin{bmatrix} -\frac{i}{4}(|u|^2+|v|^2) & \frac{i}{2}\bar{u} \\ u_x - \frac{i}{2}|v|^2 u - \frac{i}{2}v & \frac{i}{4}(|u|^2+|v|^2) \end{bmatrix}, \quad \hat{Q}_2 = -\frac{1}{2i}\begin{bmatrix} \bar{v}u & -\bar{v} \\ u + \bar{v}u^2 & -\bar{v}u \end{bmatrix}.$$

This system is useful as $z \to 0$.

# Inverse scattering

Reconstruction formula as $|z| \to \infty$:

$$\tilde{n}(x;z) = \tilde{n}_\infty(x) e_2 + \tilde{n}_\infty \bar{v}(x) e_1 z^{-1} + \mathcal{O}(z^{-2}),$$

with $\tilde{n}_\infty(x) = e^{\frac{1}{4i} \int_\infty^x (|u|^2 + |v|^2) dx}$.

Reconstruction formula as $|z| \to 0$:

$$\hat{n}(x;z) = \hat{n}_\infty(x) e_2 + \hat{n}_\infty \bar{u}(x) e_1 z + \mathcal{O}(z^2),$$

with $\hat{n}_\infty(x) = e^{-\frac{1}{4i} \int_\infty^x (|u|^2 + |v|^2) dx}$.

- Formal asymptotics (Villarroel, 1991)
- Scattering to zero with PDE analysis (Candy–Lindblad, 2016)
- Inverse scattering and the steepest descent method (Saalmann, 2017).

# Future result

Massive Thirring Model (MTM) in physical coordinates

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases}$$

## Theorem (Saalmann, 2017)

For every $u_0 \in H^{1,1}(\mathbb{R})$ sufficiently small, there exist bounded continuous functions $f_\pm$ such that for every $t \geq 1 + |x|$:

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{t-x}} \Big[ e^{i\sqrt{t^2-x^2} + i|f_+(x/t)|^2 \log(t^2-x^2)} f_+(x/t) \\ &\quad + e^{-i\sqrt{t^2-x^2} + i|f_-(x/t)|^2 \log(t^2-x^2)} f_-(x/t) \Big] + \dots \end{aligned}$$

$$\begin{aligned} v(t,x) &= \frac{1}{\sqrt{t+x}} \Big[ e^{i\sqrt{t^2-x^2} + i|f_+(x/t)|^2 \log(t^2-x^2)} f_+(x/t) \\ &\quad - e^{-i\sqrt{t^2-x^2} + i|f_-(x/t)|^2 \log(t^2-x^2)} f_-(x/t) \Big] + \dots \end{aligned}$$

# Conclusion

My talk was devoted to the massive Thirring Model in physical coordinates

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases}$$

This is an integrable case example of the massive Dirac equations in 1D:

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u, v). \end{cases}$$

The following questions were addressed:

- Global existence in the Cauchy problem
- Orbital stability of solitary waves
- Inverse scattering near zero.

# Conclusion

Most of my talk was devoted to the massive Thirring Model (MTM) in physical coordinates

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases}$$

This is an integrable case example of the massive Dirac equations in 1D:

$$\begin{cases} i(u_t + u_x) + v = \partial_{\bar{u}} W(u, v), \\ i(v_t - v_x) + u = \partial_{\bar{v}} W(u, v). \end{cases}$$

Interested in more questions?

- Semi-discretizations preserving integrability...
- Stability of $N$ solitary waves...
- Effects of nonintegrability to Dirac solitary waves...