Justification of the short-pulse equation

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References:

Yu. Liu, D.P., A. Sakovich, Dynamics of PDE 6, 291-310 (2009)
D.P., A. Sakovich, Communications in PDE 35, 613-629 (2010)
D.P., G. Schneider, submitted to SIMA (2011)

Short-pulse equation

The **short-pulse equation** is a model for propagation of ultra-short pulses with few cycles on the pulse scale [Schäfer, Wayne 2004]:

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx},$$

where all coefficients are normalized.

The short-pulse equation

- replaces the nonlinear Schrödinger equation for short wave packets
- features exact solutions for modulated pulses
- enjoys inverse scattering and an infinite set of conserved quantities

Relevant results

- T. Schafer and C.E. Wayne (2004) proved local existence in $H^2(\mathbb{R})$.
- A. Stefanov *et al.* (2010) considered a family of the generalized short-pulse equations
$$u_{xt} = u + (u^p)_{xx}$$
and proved scattering to zero for *small* initial data if $p \geq 4$.
- D.P. and A. Sakovich (2010) proved global well-posedness for *small* initial data if $p = 3$.
- Y. Liu, D.P. and A. Sakovich (2010) proved wave breaking for *large* initial data if $p = 2$ and $p = 3$.
- Remark: Global existence for *small* initial data is still opened for $p = 2$.

Integrability of the short-pulse equation

Let $x = x(y,t)$ satisfy

$$\begin{cases} x_y = \cos w, \\ x_t = -\frac{1}{2} w_t^2. \end{cases}$$

Then, $w = w(y,t)$ satisfies the sine–Gordon equation in characteristic coordinates [A. Sakovich, S. Sakovich, (2005), (2006)]:

$$w_{yt} = \sin(w).$$

Lemma

Let the mapping $[0,T] \ni t \mapsto w(\cdot,t) \in H_c^s$ *be* C^1 *and*

$$H_c^s = \left\{ w \in H^s(\mathbb{R}) : \quad \|w\|_{L^\infty} \le w_c < \frac{\pi}{2} \right\}, \quad s \ge 1.$$

Then, $x(y,t)$ *is invertible in* y *for any* $t \in [0,T]$ *and* $u(x,t) = w_t(y(x,t),t)$ *solves the short-pulse equation*

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0,T].$$

Solutions of the short-pulse equation

A kink of the sine–Gordon equation gives a *loop solution* of the short-pulse equation:

$$\left\{ \begin{array}{l} u = 2\,\mathrm{sech}(y+t), \\ x = y - 2\tanh(y+t). \end{array} \right.$$

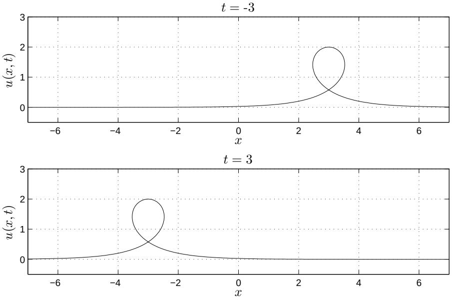

Figure: The loop solution $u(x,t)$ to the short-pulse equation

Solutions of the short-pulse equation

A breather of the sine–Gordon equation gives a *modulated pulse solution*:

$$\left\{ \begin{array}{l} u(y,t) = 4mn \dfrac{m \sin\psi \sinh\phi + n \cos\psi \cosh\phi}{m^2 \sin^2\psi + n^2 \cosh^2\phi} = u\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right), \\ x(y,t) = y + 2mn \dfrac{m \sin 2\psi - n \sinh 2\phi}{m^2 \sin^2\psi + n^2 \cosh^2\phi} = x\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right) + \frac{\pi}{m}, \end{array} \right.$$

where

$$\phi = m(y+t), \quad \psi = n(y-t), \quad n = \sqrt{1-m^2},$$

and $m \in \mathbb{R}$ is a free parameter.

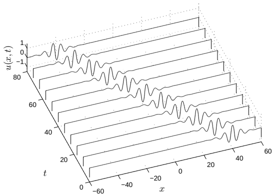

Figure: The pulse solution to the short-pulse equation with $m = 0.25$

Motivations

Nonlinear dispersive equations for short pulses have been justified in a similar context.

- D. Alterman, J. Rauch (2003) - geometric optics approach
- K. Barrailh, D. Lannes (2002); T. Colin, G. Gallice, K. Laurioux (2005) - nonlocal envelope equation with full dispersion
- M. Colin, D. Lannes (2009); D. Lannes (2011) - regularized nonlinear Schrödinger equation

For the short-pulse equation, only linearized equations were justified from Maxwell equations by using oscillatory integrals and Fourier analysis
Y. Chung, C. Jones, T. Schäfer, C.E. Wayne (2005).

Toy problem - quasilinear Klein–Gordon equation

Let us consider the quasilinear Klein–Gordon equation,

$$u_{tt} - u_{xx} + u + (u^3)_{xx} = 0.$$

Using new variables,

$$u(t,x) = 2\epsilon U(\tau,\xi), \quad \tau = \epsilon t, \quad \xi = \frac{x-t}{2\epsilon},$$

the Klein–Gordon equation can be written in the equivalent form,

$$U_{\tau\xi} = U + (U^3)_{\xi\xi} + \epsilon^2 U_{\tau\tau}.$$

The short-pulse equation appears by neglecting the last term $\epsilon^2 U_{\tau\tau}$,

$$A_{\xi\tau} = A + (A^3)_{\xi\xi}.$$

Theorem

Fix $s > \frac{7}{2}$ *and* $T > 0$. *Let* $A \in C([0,T], H^s(\mathbb{R}))$ *be a local solution of the short-pulse equation such that*

$$\sup_{\tau \in [0,T]} \|\partial_\tau^k A(\tau, \cdot)\|_{H^{s-k}} \leq \delta, \quad k = 0, 1, 2, 3,$$

for some $\delta > 0$. *Assume that there is* $\epsilon > 0$, $U_0 \in H^3(\mathbb{R})$, *and* $V_0 \in H^2(\mathbb{R})$ *such that*

$$\|U_0 - A(0, \cdot)\|_{H^2} + \|V_0 - A_\tau(0, \cdot)\|_{H^1} \leq \epsilon.$$

For a sufficiently small $\delta > 0$, *there exist* $\epsilon_0 > 0$ *and* $C_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$ *there exists a unique solution*

$$U \in C([0,T], H^3(\mathbb{R})) \cap C^1([0,T], H^2(\mathbb{R})) \cap C^2([0,T], H^1(\mathbb{R})),$$

of the Klein–Gordon equation subject to the initial data $U(0, \cdot) = U_0$, $U_\tau(0, \cdot) = V_0$ *satisfying*

$$\sup_{\tau \in [0,T]} \|U(\tau, \cdot) - A(\tau, \cdot)\|_{H^2} \leq C_0 \epsilon.$$

Assumptions of the theorem

Proposition (Schäfer & Wayne, 2004; Stefanov *et al.*, 2010)

Fix $s > \frac{3}{2}$. For any $A_0 \in H^s(\mathbb{R})$, there exists a time $\tau = \tau(\|A_0\|_{H^s}) > 0$ and a unique solution to the short-pulse equation such that

$$A \in C([0,\tau_0], H^s(\mathbb{R})) \cap C^1((0,\tau_0], H^{s-1}(\mathbb{R}))$$

and $A(0,\cdot) = A_0$. The solution depends continuously on A_0.

To obtain estimates on $\partial_\tau^k A$, we note that

$$\begin{aligned}
A_\tau &= \partial_\xi^{-1} A + (A^3)_\xi, \\
A_{\tau\tau} &= \partial_\xi^{-2} A + 3(A^2)_\xi \partial_\xi^{-1} A + 4A^3 + \frac{9}{5}(A^5)_{\xi\xi}, \\
A_{\tau\tau\tau} &= \partial_\xi^{-3} A + \partial_\xi^{-1} A^3 + 18 A^2 \partial_\xi^{-1} A + 3(A^2)_\xi \partial_\xi^{-2} A + 6 A_\xi (\partial_\xi^{-1} A)^2 \\
&\quad + \frac{27}{2}(A^4)_{\xi\xi} \partial_\xi^{-1} A + \frac{123}{5}(A^5)_\xi + \frac{27}{7}(A^7)_{\xi\xi\xi},
\end{aligned}$$

Bootstrapping of local solutions

Lemma

Let $B_0 \in L^2(\mathbb{R})$ *and consider the linear inhomogeneous equation,*

$$\left.\begin{array}{l} B_{\tau\xi} = B + F, \\ B(0,\cdot) = B_0. \end{array}\right\}$$

There exists a unique solution $B \in C([0,\tau_0], L^2(\mathbb{R}))$ *for some* $\tau_0 > 0$ *if either (a)* $F = G_\xi$ *with* $G \in C([0,\tau_0], L^2(\mathbb{R}))$ *or (b)* $F \in C^1([0,\tau_0], L^2(\mathbb{R}))$*.*

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- If $A_0 \in H^s(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$, $s > \frac{3}{2}$, then

$$\partial_\xi^{-1} A \in C([0,\tau_0], H^{s+1}(\mathbb{R})), \quad A \in C^1([0,\tau_0], H^{s-1}(\mathbb{R})).$$

- If $A_0 \in H^s(\mathbb{R}) \cap \dot{H}^{-2}(\mathbb{R})$, $s > \frac{5}{2}$, then

$$\partial_\xi^{-2} A \in C([0,\tau_0], H^{s+2}(\mathbb{R})), \quad A \in C^2([0,\tau_0], H^{s-2}(\mathbb{R})).$$

- If $A_0 \in H^s(\mathbb{R}) \cap \dot{H}^{-2}(\mathbb{R})$, $s > \frac{7}{2}$, and $\partial_\xi^{-3} A_0 + \partial_\xi^{-1} A_0^3 \in L^2(\mathbb{R})$, then

$$A \in C^3([0,\tau_0], H^{s-3}(\mathbb{R}))$$

Global well-posedness of the short-pulse equation

Proposition (D.P., A. Sakovich, 2010)

If $A_0 \in H^s(\mathbb{R})$, $s \geq 2$ and

$$\|A_0'\|_{L^2}^2 + \|A_0''\|_{L^2}^2 < \frac{1}{6},$$

there exists $C > 0$ and a unique solution $A \in C(\mathbb{R}_+, H^s(\mathbb{R}))$ of the short-pulse equation with $A(0,\cdot) = A_0$ such that $\|A(\tau,\cdot)\|_{H^s} \leq C$.

This results follows from conserved quantities [J.C. Brunelli (2005)]:

$$\cdots, E_0 = \int_{\mathbb{R}} u^2 dx, \quad E_1 = \int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} dx, \quad E_2 = \int_{\mathbb{R}} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}} dx, \cdots.$$

Starting with the quasilinear Klein–Gordon equation,

$$u_{tt} - u_{xx} + u + (u^3)_{xx} = 0.$$

we assume $\|u\|_{L^\infty} < \frac{1}{\sqrt{3}}$ and introduce

$$u_1 = u_t, \quad u_2 = (1 - 3u^2)^{1/2} u_x, \quad u_3 = u.$$

The scalar equation is equivalent to the system

$$\frac{\partial}{\partial t} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 & -(1-3u_3^2)^{1/2} & 0 \\ -(1-3u_3^2)^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{f}(\mathbf{u}).$$

Kato's theory for symmetric quasilinear systems

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Proposition (T. Kato (1975))

For any $u_0 \in H^{s+1}(\mathbb{R})$ and $v_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ such that $\|u_0\|_{L^\infty} < \frac{1}{\sqrt{3}}$, there exists a time $t_0 = t_0(\|u_0\|_{H^{s+1}} + \|v_0\|_{H^s}) > 0$ and a unique strong solution of the Klein–Gordon equation such that

$$u \in C([0,t_0], H^{s+1}(\mathbb{R})) \cap C^1([0,t_0], H^s(\mathbb{R})) \cap C^2([0,t_0], H^{s-1}(\mathbb{R})),$$

subject to $u(0,\cdot) = u_0$ and $u_t(0,\cdot) = v_0$. Moreover, the local solution depends continuously on the initial data (u_0, v_0).

Blow-up Alternative

Lemma

The local solution

$$u \in C([0,t_0), H^{s+1}(\mathbb{R})) \cap C^1([0,t_0), H^s(\mathbb{R})) \cap C^2([0,t_0), H^{s-1}(\mathbb{R})),$$

blows up in a finite time $t_0 < \infty$ *if and only if*

$$\limsup_{t\to t_0} \left(\|u(t,\cdot)\|_{L^\infty} + \|u_t(t,\cdot)\|_{L^\infty} + \|u_x(t,\cdot)\|_{L^\infty}\right) = \infty.$$

When $s = 2 > \frac{3}{2}$, the result follows from apriori estimates on the energy,

$$\begin{aligned}
E_1(u) &= \int_{\mathbb{R}} (u^2 + u_t^2 + u_x^2(1-3u^2))dx,\\
E_2(u) &= \int_{\mathbb{R}} (u_x^2 + u_{tx}^2 + u_{xx}^2(1-3u^2))dx,\\
E_3(u) &= \int_{\mathbb{R}} (u_{xx}^2 + u_{txx}^2 + u_{xxx}^2(1-3u^2))dx.
\end{aligned}$$

Flavor of the proof

For

$$E_1(u) = \int_{\mathbb{R}} (u^2 + u_t^2 + u_x^2(1 - 3u^2))dx$$

we have from the Klein–Gordon equation,

$$\frac{1}{2}\frac{dE_1(u)}{dt} = -3\int_{\mathbb{R}} u u_t u_x^2 dx, \quad t \in [0, t_0],$$

Assume that $M_{0,1,2} < \infty$, where

$$M_0 = \sup_{t\in[0,t_0]} \|u(t,\cdot)\|_{L^\infty}, \quad M_1 = \sup_{t\in[0,t_0]} \|u_t(t,\cdot)\|_{L^\infty}, \quad M_2 = \sup_{t\in[0,t_0]} \|u_x(t,\cdot)\|_{L^\infty}.$$

Then,

$$\left|\frac{dE_1(u)}{dt}\right| \leq C(M_0)M_0M_1E_1(u) \quad \Rightarrow \quad E_1(u) \leq E_1(u_0)e^{C(M_0)M_0M_1t}, \quad t \in [0, t_0],$$

hence $E_1(u)$ cannot blow up in a finite time t_0.

Reformulation in new variables

Recall that in new variables,

$$u(t,x) = 2\epsilon U(\tau,\xi), \quad \tau = \epsilon t, \quad \xi = \frac{x-t}{2\epsilon},$$

the Klein–Gordon equation can be written in the equivalent form,

$$U_{\tau\xi} = U + (U^3)_{\xi\xi} + \epsilon^2 U_{\tau\tau}.$$

Lemma

Fix $C_0 > 0$ independently of ϵ. For any $U_0 \in H^{s+1}(\mathbb{R})$ and $V_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ such that $\|U_0\|_{L^\infty} \leq C_0$, there exists an ϵ-independent time $T = T(\|U_0\|_{H^{s+1}} + \|V_0\|_{H^s}) > 0$ and a unique strong solution of the rescaled Klein-Gordon equation for any $\epsilon \neq 0$ such that

$$U(\tau,\cdot) \in C([0,\epsilon T], H^{s+1}(\mathbb{R})) \cap C^1([0,\epsilon T], H^s(\mathbb{R})) \cap C^2([0,\epsilon T], H^{s-1}(\mathbb{R})),$$

subject to $U(0,\cdot) = U_0$ and $U_\tau(0,\cdot) = V_0$. Moreover, the local solution blows up in a finite time $\tau_0 < \infty$ if and only if

$$\limsup_{\tau\to\tau_0} \left(\|U(\tau,\cdot)\|_{L^\infty} + \|U_\tau(\tau,\cdot)\|_{L^\infty} + \|U_\xi(\tau,\cdot)\|_{L^\infty}\right) = \infty.$$

Energy estimates for the error term

Setting $U = A + \epsilon R$, we obtain the Klein–Gordon equation for the error term,

$$R_{\xi\tau} = R + \epsilon^2 R_{\tau\tau} + \left(3A^2 R + 3\epsilon A R^2 + \epsilon^2 R^3\right)_{\xi\xi} + \epsilon A_{\tau\tau}.$$

We shall control the energy for the error term,

$$E = \int_{\mathbb{R}} \left(R^2 + R_\xi^2 + R_{\xi\xi}^2 + 2\epsilon^2 R_\tau^2 + \epsilon^4 R_{\tau\tau}^2\right) dx.$$

By Sobolev embedding, R and R_ξ decay to zero at infinity as $|\xi| \to \infty$ and

$$\|R\|_{L^\infty} + \|R_\xi\|_{L^\infty} \leq C E^{1/2}$$

From the Klein–Gordon equation, we also have

$$\|R_{\xi\tau}\|_{L^2} \leq C\left(\delta\epsilon + E^{1/2} + \delta^2 E^{1/2} + \delta\epsilon E + \epsilon^2 E^{3/2}\right),$$

which yields the control of $\|\epsilon R_\tau\|_{L^\infty} \leq C\left(E^{1/2} + \delta\epsilon^2 + \delta\epsilon^2 E + \epsilon^3 E^{3/2}\right)$.

Method of the proof

We have seen that the short-pulse equation has local solutions $A \in C([0,T], H^s(\mathbb{R}))$ for $T > 0$ and $s > \frac{7}{2}$ such that

$$\sup_{\tau \in [0,T]} \|\partial_\tau^k A(\tau, \cdot)\|_{H^{s-k}} \leq \delta, \quad k = 0, 1, 2, 3,$$

for some small $\delta > 0$.

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for some small $\delta > 0$.

If the initial data satisfy

$$\|U(0,\cdot) - A(0,\cdot)\|_{H^2} + \|V(0,\cdot) - A_\tau(0,\cdot)\|_{H^1} \leq \epsilon,$$

for some small $\epsilon > 0$, then

$$\|R(0,\cdot)\|_{H^2} + \|R_\tau(0,\cdot)\|_{H^1} \leq 1,$$

or $E < \infty$.

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for some small $\epsilon > 0$, then

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or $E < \infty$.

If $U(0,\cdot) \in H^3(\mathbb{R})$, and $V(0,\cdot) \in H^2(\mathbb{R})$, then there exists a local solution of the Klein–Gordon equation for the error term,

$$R \in C([0,\epsilon T], H^3(\mathbb{R})) \cap C^1([0,\epsilon T], H^2(\mathbb{R})) \cap C^2([0,\epsilon T], H^1(\mathbb{R}))$$

The existence interval is extended as long as R is controlled in the energy space $E(\tau) < \infty$ for $\tau \in [0,T]$.

Control of energy

Lemma

We have (roughly)

$$\frac{dE}{d\tau} = J, \quad |J| \leq C\left(\delta E^{1/2} + \delta^2 E + \delta E^{3/2} + \epsilon E^2\right),$$

for some (ϵ, δ)*-independent constant* $C > 0$*, as long as the solution remains in the function space*

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By Gronwall's inequality, we have

$$E(\tau) \le C_0(E(0) + \delta T)e^{C_1 \delta T}, \quad \tau \in [0,T],$$

which allows us to continue the solution from $[0, \epsilon T]$ to $[0, T]$.

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Thus, we have a local solution,

$$U \in C([0,T], H^3(\mathbb{R})) \cap C^1([0,T], H^2(\mathbb{R})) \cap C^2([0,T], H^1(\mathbb{R})),$$

satisfying

$$\sup_{\tau \in [0,T]} \|U(\tau, \cdot) - A(\tau, \cdot)\|_{H^2} \le C_0 \epsilon.$$

Summary

Solutions of the quasilinear Klein–Gordon equation,

$$u_{tt} - u_{xx} + u + (u^3)_{xx} = 0,$$

which are initially closer to small solutions of the short-pulse equation,

$$A_{\xi\tau} = A + (A^3)_{\xi\xi},$$

remain close to these solutions for long but finite time intervals.

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Initial proximity

$$\left\| u(0,\cdot) - 2\epsilon A\left(0, \frac{\cdot}{2\epsilon}\right) \right\|_{H^2} \leq C\epsilon^{1/2}, \quad \left\| u_t(0,\cdot) + A_\xi\left(0, \frac{\cdot}{2\epsilon}\right) \right\|_{H^1} \leq C\epsilon^{1/2},$$

implies

$$\sup_{t\in[0,T/\epsilon]} \left\| u(t,\cdot) - 2\epsilon A\left(\epsilon t, \frac{\cdot - t}{2\epsilon}\right) \right\|_{H^2} \leq C_0\epsilon^{1/2},$$

where the leading-order term is

$$\left\| \epsilon A_0\left(\frac{\cdot}{2\epsilon}\right) \right\|_{H^2} = \mathcal{O}(\epsilon^{-1/2}), \quad \left\| A_0'\left(\frac{\cdot}{2\epsilon}\right) \right\|_{H^1} = \mathcal{O}(\epsilon^{-1/2}).$$