Iteration method for nonlinear wave equations

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- *Proof of convergence*
- *Applications and numerical issues*

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Nonlinear wave equation:

$$u_t + u^p u_x + (\mathcal{L}u)_x = 0$$

- **KdV equation** $\mathcal{L} = \partial_x^2$
- **BO equation** $\mathcal{L} = \partial_x H$
- **ZK equation** $\mathcal{L} = \partial_x^2 + \partial_y^2$
- **KP equation** $\mathcal{L} = \partial_x^2 - \partial_x^{-2}\partial_y^2$

The Problem

Kadomtsev–Petviashvili (KP-I) equation

$$(u_t + 2uu_x + u_{xxx})_x = u_{yy}$$

Travelling wave solutions for $u(x, y, t) = \Phi(x - ct, y)$**:**

$$\left(c - \partial_x^2 + \partial_x^{-2}\partial_y^2\right)\Phi(x, y) = \Phi^2(x, y),$$

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such that

$$c > 0, \qquad \Phi(x, y) \in L^2(\mathbb{R}^2),$$

and

$$\int_{-\infty}^{\infty} \Phi(x, y)dx = 0, \quad \textbf{for any} \quad y \in \mathbb{R}$$

Iterative Solution

Double Fourier transform:

$$\Phi(x,y) \mapsto \hat{\Phi}(k_x,k_y) = \iint_{\mathbb{R}^2} \Phi(x,y)\, e^{-ik_x x - ik_y y} dx dy$$

Bound-state problem:

$$\left(c + k_x^2 + k_x^{-2} k_y^2\right) \hat{\Phi}(k_x,k_y) = \hat{\Phi}^2(k_x,k_y)$$

such that

$$c > 0, \qquad \hat{\Phi} \in L^2(\mathbb{R}^2), \qquad \hat{\Phi}(0,k_y) = 0.$$

Naive iteration algorithm:

$$\hat{u}_{n+1}(k_x,k_y) = \frac{\hat{u}_n^2(k_x,k_y)}{c + k_x^2 + k_x^{-2} k_y^2}$$

Bad news: the algorithms always diverges!

Solution by Petviashvili (1976)

Iterations with a stabilizing factor:

$$\hat{u}_{n+1}(k_x, k_y) = M_n^{\gamma} \frac{\widehat{u_n^2}(k_x, k_y)}{c + k_x^2 + k_x^{-2} k_y^2},$$

where

$$M_n = \frac{\iint_{\mathbb{R}^2} dk_x dk_y \, (c + k_x^2 + k_x^{-2} k_y^2) \, (\hat{u}_n)^2}{\iint_{\mathbb{R}^2} dk_x dk_y \, \hat{u}_n \, \widehat{u_n^2}(k)}$$

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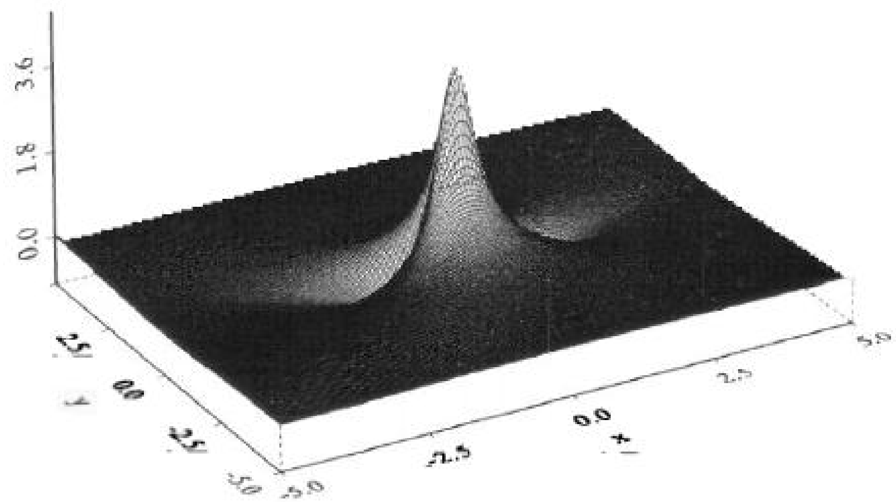
- Fixed points of iterations coincide with solutions of the problem.
- Algorithm converges if $1 < \gamma < 3$ for any $c > 0$
- Convergence is the fastest at $\gamma = 2$
- The bound state $\Phi(x, y)$ exists for any $c > 0$, such that $\Phi \in L^2(\mathbb{R}^2)$ but $\Phi \notin L^1(\mathbb{R}^2)$

Results on KP1 lumps (solitons)

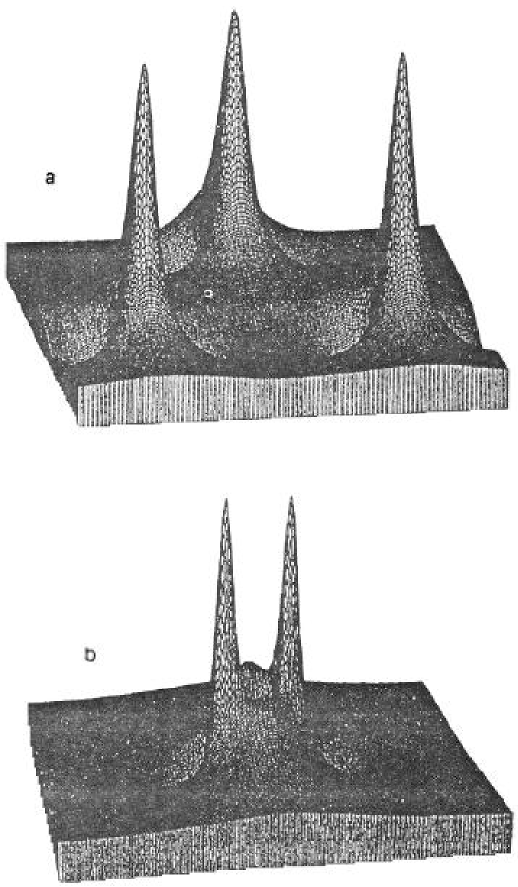
- Exact analytical expression for $\Phi(x, y)$ (Zakharov et al, 1977):

$$\Phi(x, y) = 12c\frac{3 + c^2y^2 - cx^2}{(3 + c^2y^2 + cx^2)^2}.$$

- Inverse scattering transform for KPI equation (Ablowitz, Fokas, 1983)
- Non-uniqueness of non-positive bound states $\Phi(x, y)$ (Pelinovsky, 1993)



Exact solutions of KP-I lumps



- No proof of convergence
- "Spurious" multi-humped lumps
- Applicability to other nonlinear wave equations

Convergence of a self-similar sequence

- Assume existence of a bound state $\Phi(x, y)$
- Consider a special self-similar sequence:

$$\hat{u}_n(k_x, k_y) = x_n \hat{\Phi}(k_x, k_y),$$

- x_n satisfy the power iteration map:

$$x_{n+1} = x_n^{2-\gamma}, \qquad M_n = x_n^{-1}.$$

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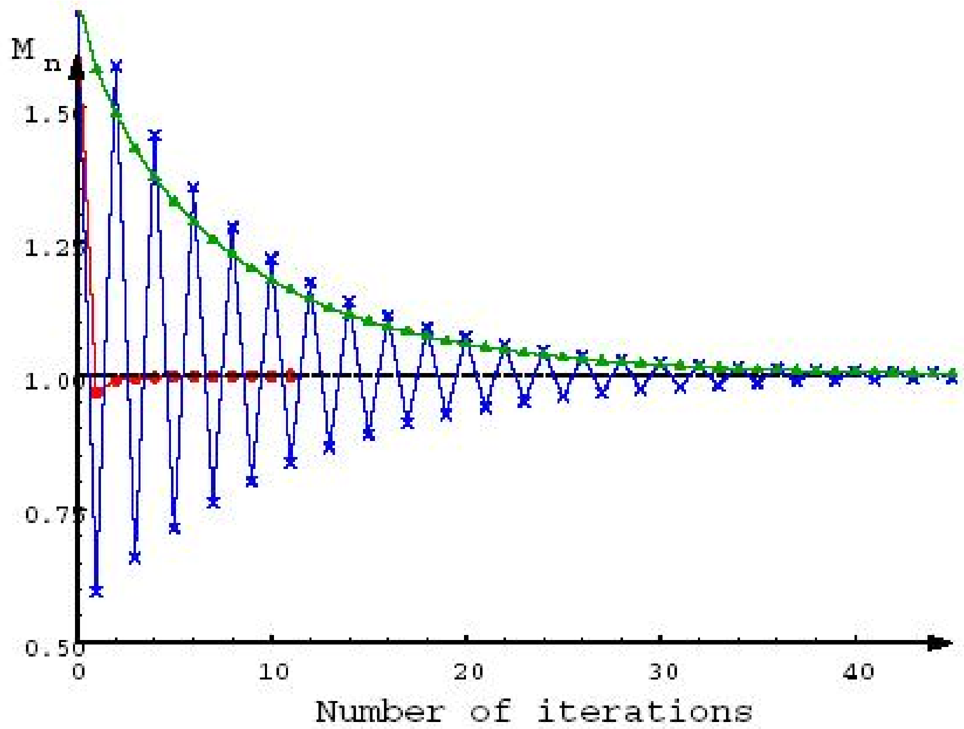
- Power iteration map converges for $1 < \gamma < 3$.
- When $\gamma = 2$, convergence occurs in a single iteration.
- There exists at least one sequence $\{x_n \hat{\Phi}(k_x, k_y)\}_{n=0}^{\infty}$, $x_0 > 0$, that converges to $\hat{\Phi}(k_x, k_y)$.

Convergence of a self-similar sequence



- $\gamma = 1.1$ - monotonic convergence
- $\gamma = 2.0$ - fastest convergence
- $\gamma = 2.9$ - sign-alternating convergence

Contraction Mapping Principle

$$\hat{u}_n \mapsto \hat{u}_{n+1} = \mathcal{A}(\hat{u}_n), \qquad u_n \in X(\mathbb{R}^2)$$

Bound state is a fixed point of $\mathcal{A}$:

$$\hat{\Phi} = \mathcal{A}(\hat{\Phi}), \qquad \Phi \in X(\mathbb{R}^2)$$

Theorem: If $\mathcal{A}(\hat{u}_n)$ has a continuous Frechet derivative $\mathcal{A}'(\hat{u}_n)$ in a small open neighborhood of $\hat{\Phi}$ in $X(\mathbb{R}^2)$ and the spectral radius of $\mathcal{A}'(\hat{\Phi})$ is smaller than one, then there is a small open ball $S(\hat{\Phi}, \delta) \in X(\mathbb{R}^2)$ such that

$$||\mathcal{A}(\hat{f}) - \mathcal{A}(\hat{g})||_{X(\mathbb{R}^2)} \leq q||\hat{f} - \hat{g}||_{X(\mathbb{R}^2)}, \qquad \forall \hat{f}, \hat{g} \in S(\hat{\Phi}, \delta)$$

where

$$q = \sup_{\hat{u}_n \in S(\hat{\Phi},\delta)} ||\mathcal{A}'(\hat{u}_n)|| < 1.$$

Frechet derivative $\mathcal{A}'(\hat{\Phi})$

- Linearize the nonlinear iteration map with

$$\hat{w}_n(k_x, k_y) = \hat{u}_n(k_x, k_y) - \hat{\Phi}(k_x, k_y),$$

and

$$m_n = M_n - 1$$

- Linearized iteration map:

$$\hat{w}_{n+1}(k_x, k_y) = \gamma m_n \hat{\Phi}(k_x, k_y) + 2\frac{\hat{\Phi}(k_x, k_y) * \hat{w}_n(k_x, k_y)}{c + k_x^2 + k_x^{-2} k_y^2},$$

such that

$$m_n = -\frac{\langle \Phi^2, w_n \rangle}{\langle \Phi^2, \Phi \rangle}$$

- Constrained function space $X_p(\mathbb{R}^2)$:

$$X_p = \{U \in X(\mathbb{R}^2) : \langle \Phi^2, U \rangle = 0\}.$$

Homogeneous linearization problem

In Fourier space,

$$\hat{q}_{n+1}(k_x, k_y) = 2\frac{\hat{\Phi}(k_x, k_y) * \hat{q}_n(k_x, k_y)}{c + k_x^2 + k_x^{-2} k_y^2}$$

In physical space,

$$q_{n+1}(x, y) = q_n(x, y) - (c + \mathcal{L})^{-1} \mathcal{H} q_n(x, y).$$

where

$$\mathcal{L} = -\partial_x^2 + \partial_x^{-2} \partial_y^2, \qquad \mathcal{H} = c + \mathcal{L} - 2\Phi(x, y)$$

Consider the generalized eigenvalue problem:

$$\mathcal{H} U = \lambda (c + \mathcal{L}) U$$

equipped with the sign-definite inner product:

$$\langle U, (c + \mathcal{L}) V \rangle$$

Decompositions and projections

- There exists a solution $U = \Phi$ for $\lambda = -1$, such that $(c + \mathcal{L})\Phi = \Phi^2$
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- There exists an orthogonal decomposition:

$$\hat{w}_n = a_n \hat{\Phi}(k_x, k_y) + \hat{q}_n,$$

  such that

  - $a_n = -m_n$
  - $m_{n+1} = (2 - \gamma) m_n$
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- $\lim_{n\to\infty} m_n = 0$ if and only if $1 < \gamma < 3$
- $\lim_{n\to\infty} q_n(x, y) = 0$ if and only if $0 < \lambda < 2$ where λ are eigenvalues of $\mathcal{H}U = \lambda \mathcal{L} U$ in $X_p(\mathbb{R}^2)$

Negative spectrum of $(c+\mathcal{L})^{-1}\mathcal{H}$

- The spectrum of $\mathcal{H}$ in $L^2(\mathbb{R}^2)$ consists of $n(\mathcal{H})$ negative eigenvalues, $z(\mathcal{H})$ zero eigenvalues, and the rest of the spectrum is bounded away of zero.

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- There exists $n(\mathcal{H})-1$ negative eigenvalues of $\mathcal{H}$ in $X_p(\mathbb{R}^2)$

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$$\forall U \in X_p(\mathbb{R}^2):$$

$$\langle U, \mathcal{H}U\rangle = \sum_{\sigma(\mathcal{H})} \mu_k \langle U_k, U_k\rangle = \sum_{\sigma((c+\mathcal{L})^{-1}\mathcal{H})} \lambda_k \langle U_k, (c+\mathcal{L})U_k\rangle$$

Positive spectrum of $(c+\mathcal{L})^{-1}\mathcal{H}$

- Equivalent form:

$$(c+\mathcal{L})U = \frac{2}{1-\lambda}\Phi(x,y)U$$

- Equivalent form:

$$\mathcal{M}V = \frac{1-\lambda}{2}V, \qquad \mathcal{M} = (c+\mathcal{L})^{-1/2}\Phi(x,y)(c+\mathcal{L})^{-1/2}$$

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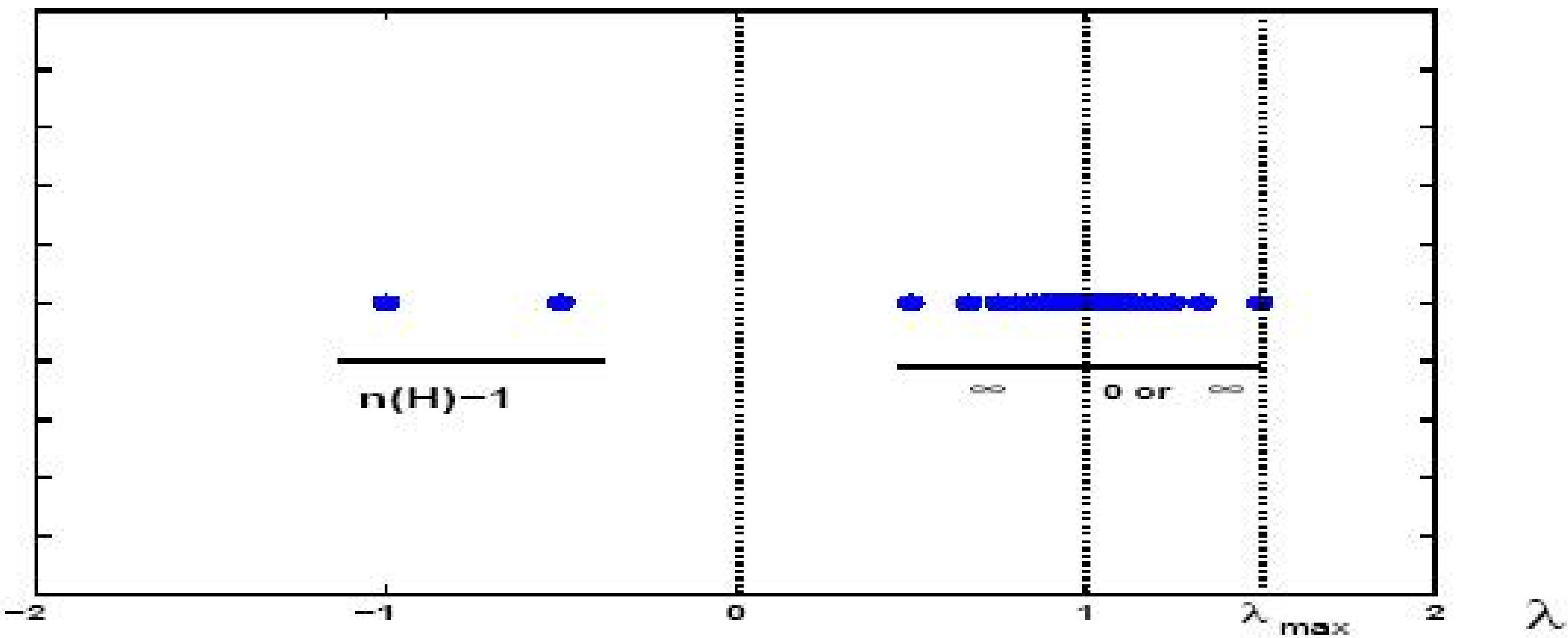
- When Φ is sign-indefinite, there exists infinitely many isolated eigenvalues λ in the interval $1 < \lambda < \lambda_{\max}$ that accumulate to $\lambda \to 1^+$

Bound on the largest eigenvalue of $(c+\mathcal{L})^{-1}\mathcal{H}$

$$\lambda = 1 - 2\frac{\langle U, \Phi U\rangle}{\langle U, (c+\mathcal{L})U\rangle}, \qquad \lambda_{\max} < 1 + \frac{2}{c}\left|\min_{(x,y)\in\mathbb{R}^2} \Phi(x,y)\right|$$



Convergence of the algorithm for KP-I equation

- $1 < \gamma < 3$ with the maximal rate at $\gamma = 2$
- $n(\mathcal{H}) = 1$
- $\lambda_{\max} < 1 + 1 = 2$, since

$$\min_{(x,y)\in\mathbb{R}^2} \Phi(x,y) = \Phi\left(\pm\frac{3}{\sqrt{c}}, 0\right) = -\frac{c}{2}.$$

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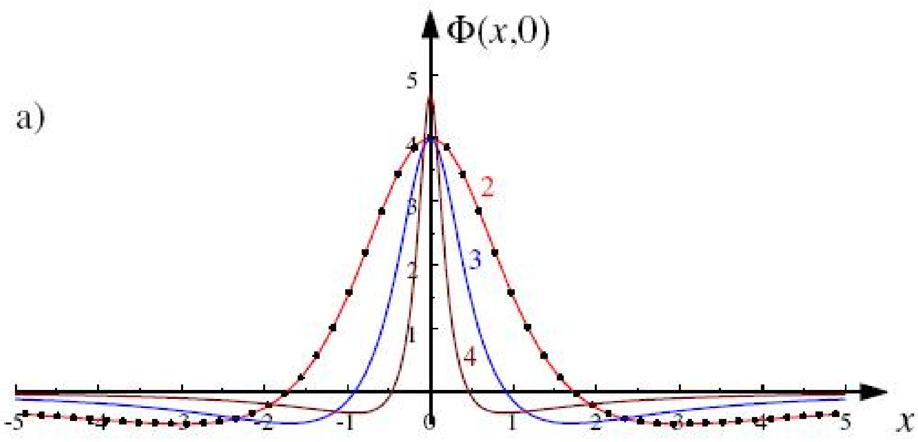
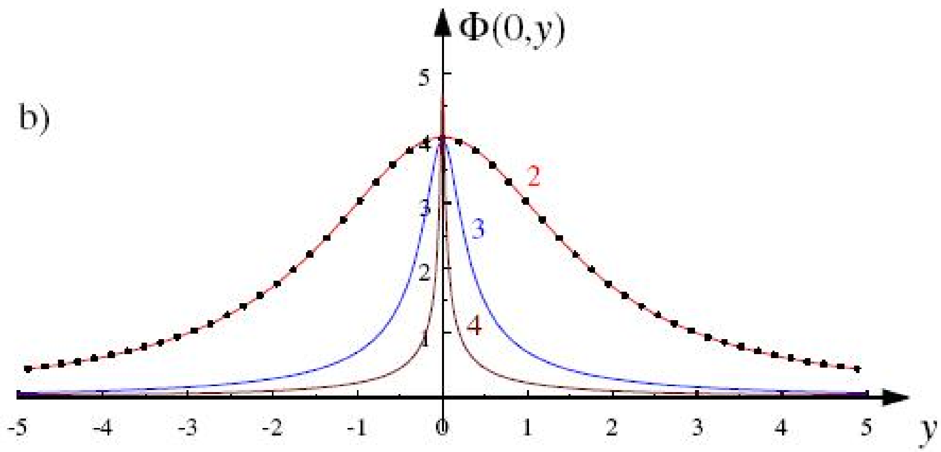
Extension to the generalized KP-I equation:

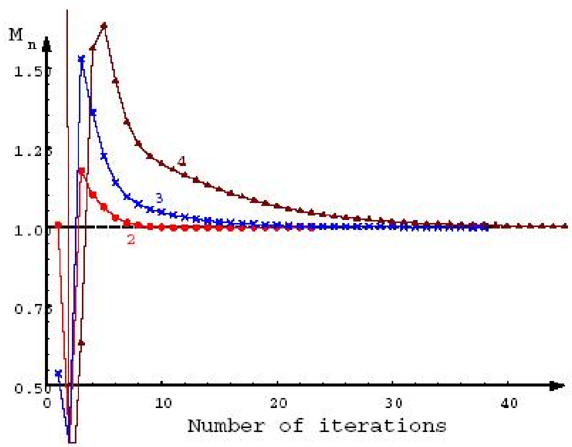
$$\left(c - \partial_x^2 + \partial_x^{-2}\partial_y^2\right)\Phi(x,y) = \Phi^p(x,y), \qquad p = 2,3,4$$

- Proof of existence for $p = 2,3,4$ by A. de Bourd, J.C.Saut (1997)
- Proof of non-existence for $p \geq 5$ by Y. Liu and X.P. Wang (1997)

Numerical solutions for $p = 2, 3, 4$







Summary

- Systematic proof of convergence of the iteration method
- Applications to classes of KdV, BO, ZK, and KP equations
- Analysis of single-humped and multi-humped non-linear waves
- Possibility of generalizations