Gaussian Solitary Waves in Granular Chains

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Canada

with R. Carles (Montpellier), E. Dumas (Grenoble),
and G. James (Grenoble)

University of Cape Town, South Africa, April 2014

Introduction

- Granular crystal chains are chains of densely packed, elastically interacting particles.
- Recent works focus on solitary and periodic travelling waves in granular chains; said to be more relevant to physical experiments.
- Periodic travelling waves in granular chains were approximated numerically and analytically
  - K.R. Jayaprakash, Yu. Starosvetsky and A.F. Vakakis, *Phys. Rev. E* **83** (2011), 036606
  - G. James, *J. Nonlinear Sci.* **22** (2012), 813
  - M. Betti and D. Pelinovsky, *J. Nonlinear Sci.* **23** (2013), 619

Experimental setups (CalTECH)

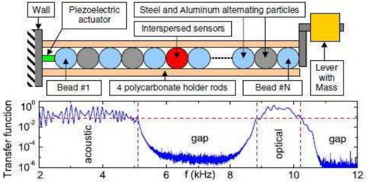

Figure : N. Boechler, G. Theocharis, S. Job, P.G. Kevrekidis, M.A. Porter, and C. Daraio, PRL **104**, 244302 (2010)

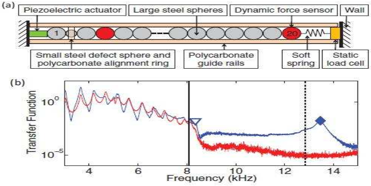

Figure : Y. Man, N. Boechler, G. Theocharis, P.G. Kevrekidis, and C. Daraio, Phys. Rev. E **85**, 037601 (2012)

On solitary travelling waves in granular chains

Proofs of existence of solitary waves were developed from the variational theory based on the differential–difference equation.

- G. Friesecke and J. Wattis, *Commun. Math. Phys.* **161** (1994), 391 - proof of existence for a general FPU lattice
- R. MacKay, *Phys. Lett. A* **251** (1999), 191 - adaptation of this method to granular chains
- J. English and R. Pego, *Proc. Amer. Math. Soc.* **133** (2005), 1763 - proof of the double-exponential tails of the solitary waves
- A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327 - proof of the bell-shaped profile of the solitary waves

The granular chain

Newton's equations define the FPU (Fermi-Pasta-Ulam) lattice:

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

where x_n is the displacement of the nth particle from a reference position versus time t.

The interaction potential for spherical beads is

$$V(x) = \frac{1}{1+\alpha} |x|^{1+\alpha} H(-x), \quad \alpha = \frac{3}{2},$$

where H is the step (Heaviside) function.

H. Hertz, *J. Reine Angewandte Mathematik* **92** (1882), 156

For the chains of hollow spherical particles of different width, we have other values of α in the range $1.2 \leq \alpha \leq 1.5$.

Travelling waves and the Boussinesq approximation

Using the relative displacements $u_n = x_n - x_{n-1}$ and applying the travelling wave reduction $u_n(t) = w_n(n-t)$, we obtain

$$\frac{d^2 w}{dz^2} = \Delta(w|w|^{\alpha-1}), \quad z \in \mathbb{R},$$

with $(\Delta w)(z) = w(z+1) - 2w(z) + w(z-1)$.

Expanding $\Delta = \partial_z^2 + \frac{1}{12}\partial_z^4$ and integrating twice, we obtain

$$w = w|w|^{\alpha-1} + \frac{1}{12}\frac{d^2}{dz^2} w|w|^{\alpha-1}, \quad z \in \mathbb{R},$$

which has compactons

$$w_{\mathrm{c}}(z) = \begin{cases} A\cos^{\frac{2}{\alpha-1}}(Bz), & |z| \leq \frac{\pi}{2B}, \\ 0, & |z| \geq \frac{\pi}{2B}, \end{cases}$$

where

$$A = \left(\frac{1+\alpha}{2\alpha}\right)^{\frac{1}{1-\alpha}}, \quad B = \frac{\sqrt{3}(\alpha-1)}{\alpha}.$$

Ill-posedness of the Boussinesq equation

The fully nonlinear Boussinesq equation takes the form

$$u_{tt} = (u|u|^{\alpha-1})_{xx} + \frac{1}{12}(u|u|^{\alpha-1})_{xxxx},$$

V.F. Nesterenko, *J. Appl. Mech. Tech. Phys.* **24** (1983), 733
K. Ahnert and A. Pikovsky, *Phys. Rev. E* **79** (2009), 026209.

Cauchy problem for the Boussinesq equation is ill-posed.

Compare with the recent work on ill-posedness of degenerate dispersive equations:

D.M. Ambrose, G. Simpson, J.D. Wright, and D.G. Yang, *Nonlinearity* **25** (2012), 2655.

Linearized Boussinesq equation

Linearizing the Boussinesq equation at the compact solution

$$u(x,t) = w(x-t) + U(x-t)e^{\lambda t},$$

we arrive at the spectral problem

$$(\lambda - \partial_z)^2 U = \left(\partial_z^2 + \frac{1}{12}\partial_z^4\right)(k_\alpha U),$$

where

$$k_\alpha(z) := \alpha w^{\alpha-1}(z) = \alpha A^{\alpha-1}\cos^2(Bz)\,\mathbf{1}_{[-\frac{\pi}{2B},\frac{\pi}{2B}]}(z).$$

The spectral problem can be closed on the compact interval $\left[-\frac{\pi}{2B},\frac{\pi}{2B}\right]$ subject to the boundary conditions

$$U\left(\pm\frac{\pi}{2B}\right) = 0, \quad U'\left(\pm\frac{\pi}{2B}\right) = 0.$$

Numerical results

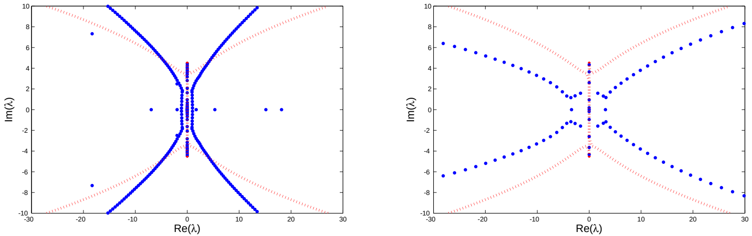

Figure : Eigenvalues of the spectral problem (blue dots) for $\alpha = 1.05$ (left) and $\alpha = 1.2$ (right). The red dotted curves show the continuous spectrum obtained in the limit case $\alpha \to 1^+$.

Korteweg–de Vries equation in the case of precompression

Consider again the FPU lattice

$$\frac{d^2 u_n}{dt^2} = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z}.$$

If $V \in C^3$ with $V''(0) = \kappa > 0$ and $V'''(0) \neq 0$, then the asymptotic multi-scale expansion

$$u_n(t) = \kappa (4V'''(0))^{-1} \varepsilon^2 y(\xi, \tau) + \text{higher order terms},$$

where $\xi := \varepsilon (n - c_s t)$, $\tau := \varepsilon^3 c_s t/24$, and $c_s := \sqrt{\kappa}$ is the "sound velocity" of linear waves, shows that y satisfies the KdV equation

$$\partial_\tau y + 3y \partial_\xi y + \partial_\xi^3 y = 0.$$

The KdV equation admits the solitary waves $y = \mathrm{sech}^2((\xi - \tau)/2)$.

Relevant results

- The KdV equation can be justified at a time scale of order ε^{-3}.
  G. Schneider and C.E. Wayne, *International Conference on Differential Equations Appl.* **5** (1998) 69
  D. Bambusi, A. Ponno, *Comm. Math. Phys.* **264** (2006), 539

- Nonlinear stability of small amplitude FPU solitons can be proved.
  G. Friesecke and R.L. Pego, *Nonlinearity* **12** (1999), 1601; **15** (2002), 1343; **17** (2004), 207; **17** (2004), 229.

- Existence and stability of *N*-soliton solutions can be proved.
  A. Hoffman and C.E. Wayne, *Nonlinearity* **21** (2008), 2911; *J. Dyn. Diff. Equat.* **21** (2009), 343.
  T. Mizumachi, *Commun. Math. Phys.* **288** (2009), 125; *SIMA* **43** (2011), 2170; *Arch. Rat. Mech. Anal.* **207** (2013), 393.

Korteweg–de Vries equation without precompression

Consider again the FPU lattice

$$\left(\frac{d^2}{dt^2} - \Delta\right) u_n = \Delta f_\alpha(u_n), \quad n \in \mathbb{Z},$$

where

$$f_\alpha(u) := u\,(|u|^{\alpha-1} - 1) = (\alpha - 1)\,u \ln|u| + \mathcal{O}((\alpha-1)^2).$$

Let $\alpha = 1 + \varepsilon^2$. Using the asymptotic multi-scale expansion

$$u_n(t) = v(\xi, \tau) + \text{higher order terms},$$

where $\xi := 2\sqrt{3}\varepsilon\,(n - t)$, $\tau := \sqrt{3}\varepsilon^3\,t$, we obtain the KdV equation with the logarithmic nonlinearity (log-KdV)

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

Stationary solutions

Stationary log-KdV equation can be integrated once to get

$$\frac{d^2 v}{d\xi^2} + v \ln |v| = 0,$$

which admits the Gaussian solitons

$$v(\xi) = \sqrt{e}\, e^{-\xi^2/4}.$$

A. Chatterjee, PRE **59** (1999), 5912

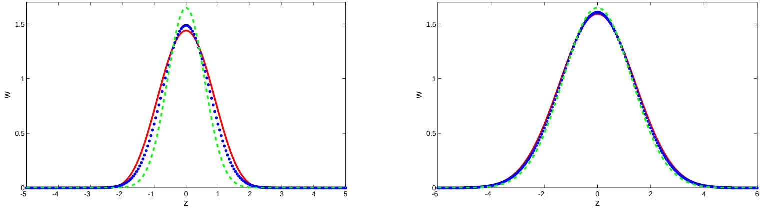

Figure : Solitary waves (blue) in comparison with the compactons (red) and the Gaussian solitons (green) for $\alpha = 1.5$ (left) and $\alpha = 1.1$ (right).

Convergence of the approximation

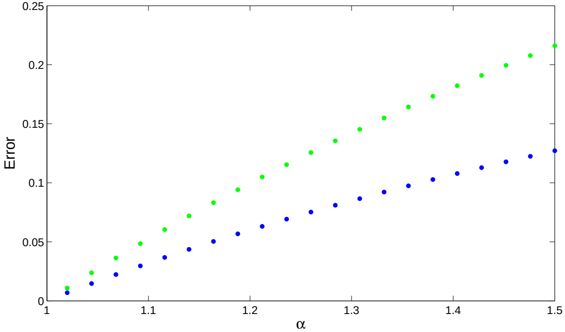

Figure : The L^∞ distance between solitary waves of the differential advance-delay equation and either the compactons (blue dots) or the Gaussian solitons (green dots) versus parameter α.

Numerical evidence of stability

Lattice of $N = 2000$ particles is excited with the initial condition of zero $x_n(0)$ and

$$\dot{x}_0(0) = 0.1, \quad \dot{x}_n(0) = 0 \text{ for all } n \geq 1.$$

A Gaussian solitary wave is formed asymptotically as t evolves.

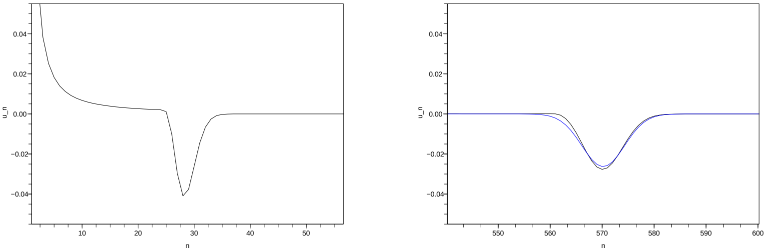

Figure : Formation of a localized wave in the Hertzian FPU lattice with $\alpha = 1.01$: left at $t \approx 30.5$, right at $t \approx 585.6$. The Gaussian approximation is shown by blue curve.

Summary of main results

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

1. Gaussian solitary wave is linearly orbitally stable in space $H^1(\mathbb{R})$.
2. For any initial data v_0 from the energy space X, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ such that the energy is not increasing in time.
3. The spectrum of the linearized operator in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ such that $0 < \omega_1 < \omega_2 < \dots$ and $\omega_n \to \infty$ as $n \to \infty$. The eigenfunctions for nonze are smooth in ξ but decay algebraically as $|\xi| \to \infty$.

Energy functional

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

can be written in the Hamiltonian form

$$\partial_\tau v = \partial_\xi E'(v),$$

where the energy functional is

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left[(\partial_\xi v)^2 - v^2 \left(\log v - \frac{1}{2} \right) \right] d\xi.$$

Gaussian wave $v_0 = e^{\frac{2-\xi^2}{4}}$ is a critical point of $E(v)$: $E'(v_0) = 0$.

Theorem 1 (G.James, D.P., 2014)

v_0 is linearly orbitally stable in space $H^1(\mathbb{R})$.

Linear operators and evolution

The Hessian operator at the critical point $v_0 = e^{\frac{2-\xi^2}{4}}$ is

$$L = E''(v_0) = -\partial_\xi^2 - 1 - \log(v_0) = -\frac{\partial^2}{\partial \xi^2} - \frac{3}{2} + \frac{\xi^2}{4}.$$

The operator L is self-adjoint in $L^2(\mathbb{R})$ with dense domain

$$D(L) = \{ u \in H^2(\mathbb{R}), \xi^2 u \in L^2(\mathbb{R}) \}.$$

The spectrum of L consists of simple eigenvalues at integers $n-1$, where $n \in \mathbb{N}_0$ (the set of natural numbers including zero).

Consider the time evolution of the perturbation u to v_0:

$$\partial_\tau u = \partial_\xi L u, \quad u(0) = u_0.$$

The solitary wave is linearly orbitally stable if for every $u_0 \in D(L)$ such that $\langle v_0, u_0 \rangle_{L^2} = 0$ there exists constant $C(u_0)$ such that

$$\|u(\tau)\|_{H^1} \leq C(u_0), \quad \tau \in \mathbb{R},$$

Symplectic decomposition

We know that $\partial_\xi L$ has a double zero eigenvalue because

$$L v_0' = 0, \quad \partial_\xi L\, v_0 = -v_0',$$

and no $u \in D(\partial_\xi L)$ exists in $\partial_\xi L u = v_0$ because $\|v_0\|_2^2 \neq 0$.

Using the decomposition

$$u(\xi,\tau) = a(\tau)\, v_0'(\xi) + b(\tau)\, v_0(\xi) + y(\xi,\tau)$$

with $\langle v_0, y\rangle_{L^2} = 0$ and $\langle \partial_\xi^{-1} v_0, y\rangle_{L^2} = 0$, we obtain

$$\frac{da}{d\tau} + b = 0, \quad \frac{db}{d\tau} = 0, \quad \frac{\partial y}{\partial \tau} = \partial_\xi L\, y.$$

If $\langle v_0, u_0\rangle_{L^2} = 0$, then $b(\tau) = b(0) = 0$ and $a(\tau) = a(0)$.

Proof of linear orbital stability

Because v_0 and v_0' are eigenvectors of L for the negative and zero eigenvalues, L is strictly positive definite on ${v_0}^{\perp} \cap {v_0'}^{\perp} \subset L^2(\mathbb{R})$.

As a result, $\|y\|_L = \langle Ly, y \rangle_{L^2}^{1/2}$ defines a norm (equivalent to a weighted H^1-norm).

From the energy balance,

$$\frac{d}{d\tau} \frac{1}{2} \|y\|_L^2 = \langle Ly, \partial_\tau y \rangle_{L^2} = \langle Ly, \partial_\xi Ly \rangle_{L^2} = 0,$$

we obtain the Lyapunov stability of the zero equilibrium $y = 0$ in the constrained space $\langle v_0, y \rangle_{L^2} = 0$ and $\langle \partial_\xi^{-1} v_0, y \rangle_{L^2} = 0$. $\square$

The constrained space corresponds to the modulation of the two parameters of the Gaussian solitary wave.

Global existence of solutions

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0$$

has the associated energy functional

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left[(\partial_\xi v)^2 - v^2 \left(\log v - \frac{1}{2} \right) \right] d\xi,$$

defined in the function space

$$X := \left\{ v \in H^1(\mathbb{R}) : \quad v^2 \log|v| \in L^1(\mathbb{R}) \right\}.$$

Theorem 2 (R. Carles, D.P., 2014)

For any $v_0 \in X$, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation such that

$$\|v(\tau)\|_{L^2} \leq \|v_0\|_{L^2}, \quad E(v(\tau)) \leq E(v_0), \quad \text{for all } \tau \in \mathbb{R}.$$

Proof of global existence

1. Construct an approximation of the logarithmic nonlinearity (Cazenave, 1980):

$$f_\varepsilon(v) = \begin{cases} v\log(v), & |v| \geq \varepsilon, \\ \left(\log(\varepsilon) - \frac{3}{4}\right) v + \frac{1}{\varepsilon^2} v^3 - \frac{1}{4\varepsilon^4} v^5, & |v| \leq \varepsilon, \end{cases}$$

hence $f_\varepsilon \in C^2(\mathbb{R})$ and $f_\varepsilon(v) \to v\log(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$.

Proof of global existence

1. Construct an approximation of the logarithmic nonlinearity (Cazenave, 1980):

$$f_\varepsilon(v) = \begin{cases} v\log(v), & |v| \geq \varepsilon, \\ \left(\log(\varepsilon) - \frac{3}{4}\right) v + \frac{1}{\varepsilon^2} v^3 - \frac{1}{4\varepsilon^4} v^5, & |v| \leq \varepsilon, \end{cases}$$

   hence $f_\varepsilon \in C^2(\mathbb{R})$ and $f_\varepsilon(v) \to v\log(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$.

2. Obtain existence of the global approximating solutions $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$ of the generalized KdV equations

$$\begin{cases} v^\varepsilon_\tau + v^\varepsilon_{\xi\xi\xi} + f'_\varepsilon(v^\varepsilon) v^\varepsilon_\xi = 0, & \tau > 0, \\ v^\varepsilon|_{\tau=0} = v_0. \end{cases}$$

   (Kenig, Ponce, Vega, 1991).

Proof of global existence

1. Construct an approximation of the logarithmic nonlinearity (Cazenave, 1980):

$$f_\varepsilon(v) = \begin{cases} v\log(v), & |v| \geq \varepsilon, \\ \left(\log(\varepsilon) - \frac{3}{4}\right) v + \frac{1}{\varepsilon^2} v^3 - \frac{1}{4\varepsilon^4} v^5, & |v| \leq \varepsilon, \end{cases}$$

   hence $f_\varepsilon \in C^2(\mathbb{R})$ and $f_\varepsilon(v) \to v\log(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$.
2. Obtain existence of the global approximating solutions $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$ of the generalized KdV equations

$$\begin{cases} v^\varepsilon_\tau + v^\varepsilon_{\xi\xi\xi} + f'_\varepsilon(v^\varepsilon) v^\varepsilon_\xi = 0, & \tau > 0, \\ v^\varepsilon|_{\tau=0} = v_0. \end{cases}$$

   (Kenig, Ponce, Vega, 1991).
3. Obtain uniform estimates for all $\varepsilon > 0$ and $\tau \in \mathbb{R}$:

$$\|v^\varepsilon(\tau)\|_{H^1} + \|(v^\varepsilon(\tau))^2 \log(v^\varepsilon(\tau))\|_{L^1} \leq C(v_0).$$

4. Pass to the limit $\varepsilon \to 0$ and obtain a global solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation. □

Uniqueness and global well-posedness

Lemma: *Assume that a solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation satisfies the additional condition*

$$(\log |v|)_\xi \in L^\infty([-\tau_0, \tau_0] \times \mathbb{R}).$$

Then, the solution v is unique for every $\tau \in (-\tau_0, \tau_0)$, depends continuously on the initial data $v_0 \in X$, and satisfies $\|v(\tau)\|_{L^2} = \|v_0\|_{L^2}$ and $E(v(\tau)) = E(v_0)$ for all $\tau \in (-\tau_0, \tau_0)$.

Uniqueness and global well-posedness

Lemma: *Assume that a solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation satisfies the additional condition*

$$(\log|v|)_\xi \in L^\infty([-\tau_0, \tau_0] \times \mathbb{R}).$$

Then, the solution v is unique for every $\tau \in (-\tau_0, \tau_0)$, depends continuously on the initial data $v_0 \in X$, and satisfies $\|v(\tau)\|_{L^2} = \|v_0\|_{L^2}$ and $E(v(\tau)) = E(v_0)$ for all $\tau \in (-\tau_0, \tau_0)$.

- $\partial_\xi \log|v|$ is unbounded as $|\xi| \to \infty$ for the Gaussian solitary wave.
- Nonlinear orbital stability of Gaussian solitary wave is conditional that the global solution $v \in L^\infty(\mathbb{R}, X)$ is unique and depends continuously on the initial data $v_0 \in X$.

Spectral stability

If $v = V(\xi)e^{\lambda\tau}$, we arrive to the linear eigenvalue problem

$$\partial_\xi L V = \lambda V.$$

Under the properties of L ($\sigma(L) = \{n-1, \quad n \in \mathbb{N}_0\}$), spectral stability of the Gaussian wave v_0 follows from an adaptation of recent works:

- T. Kapitula, A. Stefanov, *Stud. Appl. Math.* (2014).
- D.P., in *Spectral analysis, stability, and bifurcation in modern nonlinear physical systems* (Wiley–ISTE, 2014).

Theorem 3 (R. Carles, D.P., 2014)

The spectrum of $\partial_x L$ in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ such that $0 < \omega_1 < \omega_2 < \dots$ and $\omega_n \to \infty$ as $n \to \infty$. The eigenfunctions for nonzero eigenvalues are smooth in ξ but decay algebraically as $|\xi| \to \infty$.

Further remarks

- Because the spectrum of $\partial_x L$ is purely discrete, no asymptotic stability result can hold for Gaussian solitary waves.
- This agrees with the result of Cazenave for the log–NLS equation: the L^p norms at the solution v for any $p \geq 2$ including $p = \infty$ may not vanish as $t \to \infty$ (or in a finite time).

Further remarks

- Because the spectrum of $\partial_x L$ is purely discrete, no asymptotic stability result can hold for Gaussian solitary waves.
- This agrees with the result of Cazenave for the log–NLS equation: the L^p norms at the solution v for any $p \geq 2$ including $p = \infty$ may not vanish as $t \to \infty$ (or in a finite time).
- Nonlinear analysis of perturbations to the Gaussian solitary wave becomes now problematic. If $v(\xi, \tau) := v_0(\xi) + w(\xi, \tau)$ is set, then w satisfies

$$w_\tau = \partial_\xi L w - \partial_\xi N(w),$$

where

$$N(w) := w \log\left(1 + \frac{w}{v_0}\right) + v_0 \left[\log\left(1 + \frac{w}{v_0}\right) - \frac{w}{v_0}\right].$$

However, w/v_0 may grow like an inverse Gaussian function of ξ.

Proof of spectral stability

The linear eigenvalue problem

$$AV = \lambda V, \quad A := \partial_\xi L = -\partial_\xi^3 + \frac{1}{4}(\xi^2 - 6)\partial_\xi + \frac{1}{2}\xi,$$

can be written in the equivalent form with the Fourier transform

$$\hat{A}\hat{V} = \lambda \hat{V}, \quad \hat{A} = \frac{i}{4} k \left(-\partial_k^2 + 4k^2 - 6\right).$$

with the natural choice $\lambda = \frac{i}{4}E$.

Eigenfunctions of A are defined in the domain $X_A := D(A) \cap \dot{H}^{-1}(\mathbb{R})$,

$$D(A) = \left\{u \in H^3(\mathbb{R}) : \quad \xi^2 \partial_\xi u \in L^2(\mathbb{R}), \quad \xi u \in L^2(\mathbb{R})\right\}.$$

In the Fourier form, the domain X_A becomes

$$\hat{X}_A = \left\{\hat{u} \in H^1(\mathbb{R}) : \quad k\partial_k^2 \hat{u} \in L^2(\mathbb{R}), \quad k^3 \hat{u} \in L^2(\mathbb{R}), \quad k^{-1}\hat{u} \in L^2(\mathbb{R})\right\}.$$

Proof of spectral stability

The linear eigenvalue problem is

$$\frac{d^2\hat{u}}{dk^2} + \left(\frac{E}{k} + 6 - 4k^2\right)\hat{u}(k) = 0, \quad k \in \mathbb{R}.$$

- As $k \to 0$, two linearly independent solutions exist

$$\hat{u}_1(k) = k + \mathcal{O}(k^2), \quad \hat{u}_2(k) = 1 + \mathcal{O}(k\log(k)).$$

  The second solution does not belong to $\hat{X}_A$.

- As $|k| \to \infty$, the decaying solution satisfies

$$\hat{u}(k) = ke^{-k^2}\left(1 + \mathcal{O}(|k|^{-1})\right).$$

  The shooting problem is over-determined.

Proof of spectral stability

- The way around is the weak piecewise definition of the eigenfunction:

$$\hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

Proof of spectral stability

- The way around is the weak piecewise definition of the eigenfunction:

$$\hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

- For $\hat{u}_+$, we set $\hat{u}_+(k) = k^{1/2}\hat{v}_+(k)$ and obtain

$$k^{1/2}\left(-\frac{d^2}{dk^2} + 4k^2 - 6\right)k^{1/2}\hat{v}_+(k) = E\hat{v}_+(k), \quad k \in (0,\infty),$$

which is now in the symmetric form. Hence $E \in \mathbb{R}$.

Proof of spectral stability

- The way around is the weak piecewise definition of the eigenfunction:

$$\hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

- For $\hat{u}_+$, we set $\hat{u}_+(k) = k^{1/2}\hat{v}_+(k)$ and obtain

$$k^{1/2}\left(-\frac{d^2}{dk^2} + 4k^2 - 6\right)k^{1/2}\hat{v}_+(k) = E\hat{v}_+(k), \quad k \in (0,\infty),$$

which is now in the symmetric form. Hence $E \in \mathbb{R}$.

- For $E = 0$, we have $\hat{v}_+ = k^{1/2}e^{-k^2} > 0$ for $k > 0$. By Sturm's Theorem, the set of eigenvalues $\{E_n\}_{n\in\mathbb{N}_0}$ satisfies $0 = E_0 < E_1 < E_2 < \dots$ and $E_n \to \infty$ as $n \to \infty$. □

Numerical illustration

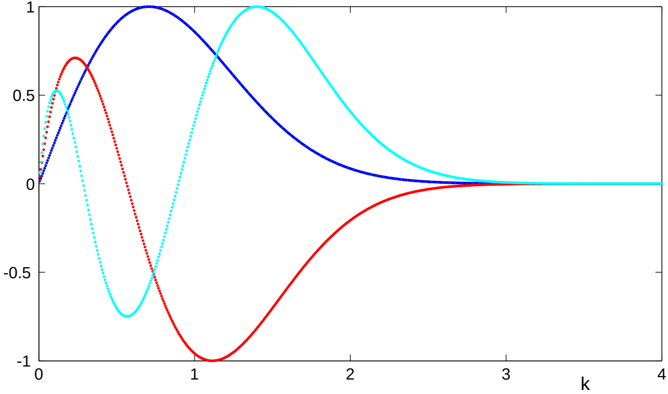

Figure : Eigenfunctions $\hat{u}$ of the spectral problem versus k for the first three eigenvalues $E_0 = 0$, $E_1 \approx 5.411$, and $E_2 \approx 12.308$.

Further development - justification of convergence

Writing the differential advance-delay equation

$$\frac{d^2 v}{dz^2} = \Delta v^{1+\varepsilon^2}, \quad z \in \mathbb{R},$$

the equivalent integral Fourier form, we obtain a fixed-point problem

$$\hat{v}(k) = \frac{4}{k^2}\sin^2\left(\frac{k}{2}\right)\widehat{v^{1+\varepsilon^2}}(k). \quad k \in \mathbb{R},$$

Expansion near $k = 0$ yields the stationary log–KdV equation

$$0 = -\frac{k^2}{12}\hat{v}(k) + \varepsilon^2 \widehat{v\log(v)}(k), \quad \text{as} \quad k \to 0.$$

Consider now solitary waves such that $v(z) \geq v_0 > 0$ for all $z \in \mathbb{R}$.

Theorem 4 (E. Dumas, D.P., 2014)

For sufficiently small ε, there exists a solution v in $H^1(\mathbb{R})$ near the solitary wave v_0 such that

$$\sup_{z\in\mathbb{R}} |v(z) - v_0(z)| \leq C_0 \varepsilon^{1/6}.$$

Further development - the KdV equation with compactons

Beyond order of $(\alpha-1)^2=\varepsilon^4$, we can rewrite the nonlinearity of the differential advance-delay equation

$$\left(\frac{d^2}{dt^2}-\Delta\right)u_n=\Delta f_\alpha(u_n), \quad n\in\mathbb{Z},$$

in the equivalent form:

$$\begin{aligned} f_\alpha(u) &:= u(|u|^{\alpha-1}-1)=(\alpha-1)\,u\ln|u|+\mathcal{O}((\alpha-1)^2) \\ &= \alpha\left(u-u|u|^{\frac{1}{\alpha}-1}\right)+\mathcal{O}((\alpha-1)^2). \end{aligned}$$

Consequently, we can derive the generalized KdV equation

$$\partial_\tau v+\partial_\xi^3 v+\frac{\alpha}{\alpha-1}\partial_\xi(v-v|v|^{\frac{1}{\alpha}-1})=0$$

at the same order as the log-KdV equation. The generalized KdV equation has exact compacton solutions.

Open questions

- Convergence of Gaussian waves and compactons in the generalized KdV equation to the solitary wave in the FPU chains.
- Orbital stability of Gaussian waves or compactons in the log-KdV and the generalized KdV equations.
- Transfer of orbital stability results to the solitary waves in the FPU chains with Hertzian potentials.
- Development of numerical methods for the log–KdV and generalized KdV equations.

References

- G. James and D.E. Pelinovsky, Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattices with Hertzian potentials, *Proceedings of Royal Society A* **470**, 20130465 (20 pages) (2014).

- R. Carles and D.E. Pelinovsky, On the orbital stability of Gaussian solitary waves in the log-KdV equation, arXiv:1401.1738 (2014)

- E. Dumas and D.E. Pelinovsky, Justification of the log-KdV equation in granular chains : the case of precompression, to be submitted (2014).

References

- G. James and D.E. Pelinovsky, Gaussian solitary waves and compactons in Fermi-Pasta-Ulam lattices with Hertzian potentials, *Proceedings of Royal Society A* **470**, 20130465 (20 pages) (2014).
- R. Carles and D.E. Pelinovsky, On the orbital stability of Gaussian solitary waves in the log-KdV equation, arXiv:1401.1738 (2014)
- E. Dumas and D.E. Pelinovsky, Justification of the log-KdV equation in granular chains : the case of precompression, to be submitted (2014).

Merci beaucoup pour votre attention!