On the orbital stability of Gaussian solitary waves in granular chains

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Canada

with R. Carles (Montpellier) and G. James (Grenoble)

Workshop "Stability of solitary waves", Pisa, Italy, May 26-30, 2014

Introduction

- Granular chains contain densely packed, elastically interacting particles with Hertzian contact forces.

Introduction

- Granular chains contain densely packed, elastically interacting particles with Hertzian contact forces.
- Existence of solitary waves was considered previously within the variational theory:
  - G. Friesecke and J. Wattis, *Commun. Math. Phys.* **161** (1994), 391 - proof of existence for a general FPU lattice
  - R. MacKay, *Phys. Lett. A* **251** (1999), 191 - adaptation of this method to granular chains
  - J. English and R. Pego, *Proc. Amer. Math. Soc.* **133** (2005), 1763 - proof of the double-exponential tails of the solitary waves
  - A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327 - proof of the bell-shaped profile of the solitary waves

Introduction

- Granular chains contain densely packed, elastically interacting particles with Hertzian contact forces.
- Existence of solitary waves was considered previously within the variational theory:
  - G. Friesecke and J. Wattis, *Commun. Math. Phys.* **161** (1994), 391 - proof of existence for a general FPU lattice
  - R. MacKay, *Phys. Lett. A* **251** (1999), 191 - adaptation of this method to granular chains
  - J. English and R. Pego, *Proc. Amer. Math. Soc.* **133** (2005), 1763 - proof of the double-exponential tails of the solitary waves
  - A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327 - proof of the bell-shaped profile of the solitary waves
- We consider solitary waves by simplifying the Fermi-Pasta-Ulam lattice to a Korteweg-de Vries equation.

The Fermi-Pasta-Ulam granular chain

Newton's equations define the FPU (Fermi-Pasta-Ulam) lattice:

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

where x_n is the displacement of the nth particle from a reference position versus time t.

The interaction potential for spherical beads is

$$V(x) = \frac{1}{1+\alpha} |x|^{1+\alpha} H(-x), \quad \alpha = \frac{3}{2},$$

where H is the step (Heaviside) function.

H. Hertz, *J. Reine Angewandte Mathematik* **92** (1882), 156

For the chains of hollow spherical particles of different width, we have other values of α in the range $1.2 \leq \alpha \leq 1.5$.

The logarithmic Korteweg–de Vries equation

Consider the FPU lattice for relative displacements $u_n := x_{n+1} - x_n$,

$$\left(\frac{d^2}{dt^2} - \Delta\right) u_n = \Delta f_\alpha(u_n), \quad n \in \mathbb{Z},$$

where

$$f_\alpha(u) := u\left(|u|^{\alpha-1} - 1\right) = (\alpha - 1)\, u \ln|u| + \mathcal{O}((\alpha-1)^2).$$

Boussinesq approximations with compactly supported solitary waves are ill-posed and cannot be justified.

V.F. Nesterenko, *J. Appl. Mech. Tech. Phys.* **24** (1983), 733

The logarithmic Korteweg–de Vries equation

Consider the FPU lattice for relative displacements $u_n := x_{n+1} - x_n$,

$$\left(\frac{d^2}{dt^2} - \Delta\right) u_n = \Delta f_\alpha(u_n), \quad n \in \mathbb{Z},$$

where

$$f_\alpha(u) := u(|u|^{\alpha-1} - 1) = (\alpha - 1)\, u \ln|u| + \mathcal{O}((\alpha-1)^2).$$

Boussinesq approximations with compactly supported solitary waves are ill-posed and cannot be justified.
V.F. Nesterenko, *J. Appl. Mech. Tech. Phys.* **24** (1983), 733

To consider the limit $\alpha \to 1$, we set $\alpha = 1 + \varepsilon^2$ and use the asymptotic multi-scale expansion

$$u_n(t) = v(\xi, \tau) + \text{higher order terms},$$

where $\xi := 2\sqrt{3}\varepsilon(n - t)$ and $\tau := \sqrt{3}\varepsilon^3 t$. At $\mathcal{O}(\varepsilon^4)$, we obtain the KdV equation with the logarithmic nonlinearity (log-KdV)

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

Korteweg–de Vries equation for regular FPU lattices

If $V \in C^3$ with $V''(0) > 0$ and $V'''(0) \neq 0$, the same expansion reduces the FPU lattice to the quadratic KdV equation

$$\partial_\tau v + v\partial_\xi v + \partial_\xi^3 v = 0.$$

The KdV equation admits the solitary waves $v \sim \mathrm{sech}^2(\xi - c\tau)$.

- The KdV equation can be justified at a time scale of order ε^{-3}.
  G. Schneider–C.E. Wayne (2000); D. Bambusi–A. Ponno (2006).

- Nonlinear stability of small amplitude FPU solitons can be proved.
  G. Friesecke–R.L. Pego (1999-2004).

- Existence and stability of N-soliton solutions can be proved.
  A. Hoffman–C.E. Wayne (2008); T. Mizumachi (2012).

Stationary solutions

Stationary log-KdV equation can be integrated once to get

$$\frac{d^2 v}{d\xi^2} + v \log|v| = 0,$$

which admits the Gaussian solitons

$$v(\xi) = \sqrt{e}\, e^{-\xi^2/4}.$$

A. Chatterjee, PRE **59** (1999), 5912;

G. James–D.P., Proc. Roy. Soc. A **470** (2014), 20130465.

Stationary solutions

Stationary log-KdV equation can be integrated once to get

$$\frac{d^2 v}{d\xi^2} + v \log|v| = 0,$$

which admits the Gaussian solitons

$$v(\xi) = \sqrt{e}\, e^{-\xi^2/4}.$$

A. Chatterjee, PRE **59** (1999), 5912;

G. James–D.P., Proc. Roy. Soc. A **470** (2014), 20130465.

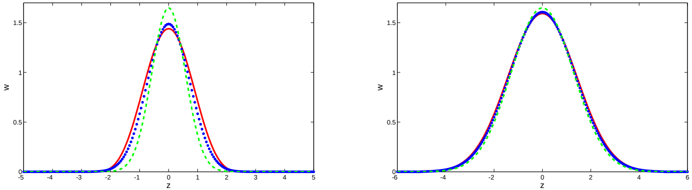

Figure : Solitary waves of the FPU chain (blue), Nesterenko compactons (red) and Gaussian solitons (green) for $\alpha = 1.5$ (left) and $\alpha = 1.1$ (right).

Numerical evidence of convergence of the approximation

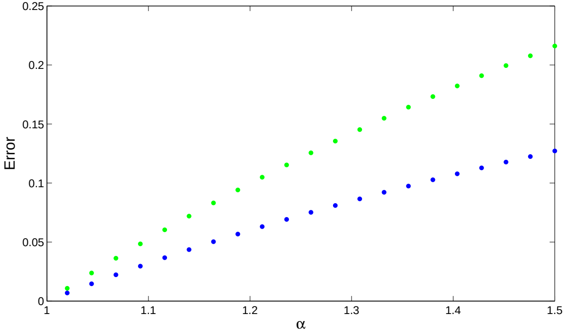

Figure : The L^∞ distance between solitary waves of the FPU chain and either Nesterenko compactons (blue dots) or Gaussian solitons (green dots) vs. α.

Numerical evidence of stability

Lattice of $N = 2000$ particles is excited with the initial condition of zero $x_n(0)$ and

$$\dot{x}_0(0) = 0.1, \quad \dot{x}_n(0) = 0 \text{ for all } n \geq 1.$$

A Gaussian solitary wave is formed asymptotically as t evolves.

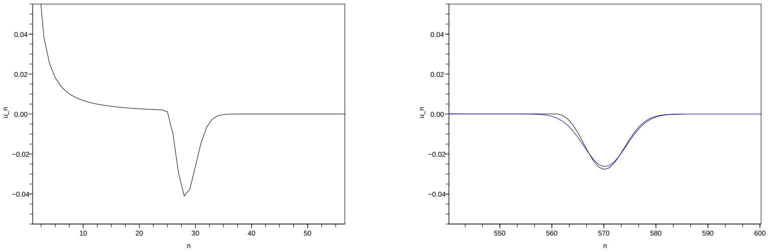

Figure : Formation of a localized wave in the Hertzian FPU lattice with $\alpha = 1.01$: left at $t \approx 30.5$, right at $t \approx 585.6$. The Gaussian wave is shown by blue curve.

Main results

The log-KdV equation

$$\partial_t v + \partial_x (v \log |v|) + \partial_x^3 v = 0.$$

R. Carles–D.P., *Nonlinearity*, submitted (2014).

1. For any initial data v_0 from the energy space X, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ s.t. the energy is not increasing.

2. The spectrum of the linearized operator in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ s.t. $0 < \omega_1 < \omega_2 < \dots$ and $\omega_n \to \infty$ as $n \to \infty$. Eigenfunctions for nonzero eigenvalues are smooth in x but decay algebraically as $|x| \to \infty$.

3. Gaussian solitary wave is linearly orbitally stable in space $H^1(\mathbb{R})$.

Global existence of solutions

The log-KdV equation can be written in the Hamiltonian form

$$\partial_t v = \partial_x E'(v),$$

where the energy functional is

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left[(\partial_x v)^2 - v^2 \left(\log|v| - \frac{1}{2} \right) \right] dx,$$

defined in the function space

$$X := \left\{ v \in H^1(\mathbb{R}) : \quad v^2 \log|v| \in L^1(\mathbb{R}) \right\}.$$

Global existence of solutions

The log-KdV equation can be written in the Hamiltonian form

$$\partial_t v = \partial_x E'(v),$$

where the energy functional is

$$E(v) = \frac{1}{2}\int_{\mathbb{R}} \left[(\partial_x v)^2 - v^2 \left(\log|v| - \frac{1}{2} \right) \right] dx,$$

defined in the function space

$$X := \left\{ v \in H^1(\mathbb{R}) : \quad v^2 \log|v| \in L^1(\mathbb{R}) \right\}.$$

Theorem 1 (R. Carles–D.P., 2014)

For any $v_0 \in X$, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation such that

$$\|v(\tau)\|_{L^2} \leq \|v_0\|_{L^2}, \quad E(v(\tau)) \leq E(v_0), \quad \text{for all } \tau \in \mathbb{R}.$$

Step 1: approximating solutions

- Construct an approximation of the logarithmic nonlinearity (Cazenave, 1980):

$$f_\varepsilon(v) = \begin{cases} v\log(|v|), & |v| \geq \varepsilon, \\ \left(\log(\varepsilon) - \frac{3}{4}\right) v + \frac{1}{\varepsilon^2} v^3 - \frac{1}{4\varepsilon^4} v^5, & |v| \leq \varepsilon, \end{cases}$$

hence $f_\varepsilon \in C^2(\mathbb{R})$ and $f_\varepsilon(v) \to v\log(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$.

Step 1: approximating solutions

- Construct an approximation of the logarithmic nonlinearity (Cazenave, 1980):

$$f_\varepsilon(v) = \begin{cases} v \log(|v|), & |v| \geq \varepsilon, \\ \left(\log(\varepsilon) - \frac{3}{4}\right) v + \frac{1}{\varepsilon^2} v^3 - \frac{1}{4\varepsilon^4} v^5, & |v| \leq \varepsilon, \end{cases}$$

hence $f_\varepsilon \in C^2(\mathbb{R})$ and $f_\varepsilon(v) \to v \log(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$.

- For a given initial data $v_0 \in H^1(\mathbb{R})$, obtain a sequence of the global approximating solutions $v^\varepsilon \in C([-T_\varepsilon, T_\varepsilon], H^1(\mathbb{R}))$ of the generalized KdV equations

$$\begin{cases} v_t^\varepsilon + v_{xxx}^\varepsilon + f_\varepsilon'(v^\varepsilon) v_x^\varepsilon = 0, & t > 0, \\ v^\varepsilon|_{t=0} = v_0. \end{cases}$$

(Kenig, Ponce, Vega, 1991).

Step 1: approximating solutions

- Construct an approximation of the logarithmic nonlinearity (Cazenave, 1980):

$$f_\varepsilon(v) = \begin{cases} v\log(|v|), & |v| \geq \varepsilon, \\ \left(\log(\varepsilon) - \frac{3}{4}\right)v + \frac{1}{\varepsilon^2}v^3 - \frac{1}{4\varepsilon^4}v^5, & |v| \leq \varepsilon, \end{cases}$$

hence $f_\varepsilon \in C^2(\mathbb{R})$ and $f_\varepsilon(v) \to v\log(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$.

- For a given initial data $v_0 \in H^1(\mathbb{R})$, obtain a sequence of the global approximating solutions $v^\varepsilon \in C([-T_\varepsilon, T_\varepsilon], H^1(\mathbb{R}))$ of the generalized KdV equations

$$\begin{cases} v_t^\varepsilon + v_{xxx}^\varepsilon + f_\varepsilon'(v^\varepsilon)v_x^\varepsilon = 0, & t > 0, \\ v^\varepsilon|_{t=0} = v_0. \end{cases}$$

(Kenig, Ponce, Vega, 1991).

- **Remark:** $T_\varepsilon \to 0$ as $\varepsilon \to 0$ because $f(v) = v\log(v) \notin C^2(\mathbb{R})$.

Step 2: uniform energy estimates

- Use energy conservation

$$\|v^{\varepsilon}(t)\|_{L^2} = \|v_0\|_{L^2}, \quad E_{\varepsilon}(v^{\varepsilon}(t)) = E_{\varepsilon}(v_0), \quad \text{for} \quad \text{every } t \in [-T_{\varepsilon}, T_{\varepsilon}],$$

where

$$E_{\varepsilon}(v) := \frac{1}{2}\int_{\mathbb{R}} (v_x)^2 dx - \int_{\mathbb{R}} W_{\varepsilon}(v) dx, \quad W_{\varepsilon}(v) := \int_0^v f_{\varepsilon}(v) dv.$$

Step 2: uniform energy estimates

- Use energy conservation

$$\|v^\varepsilon(t)\|_{L^2} = \|v_0\|_{L^2}, \quad E_\varepsilon(v^\varepsilon(t)) = E_\varepsilon(v_0), \quad \text{for} \quad \text{every } t \in [-T_\varepsilon, T_\varepsilon],$$

where

$$E_\varepsilon(v) := \frac{1}{2}\int_{\mathbb{R}} (v_x)^2 dx - \int_{\mathbb{R}} W_\varepsilon(v) dx, \quad W_\varepsilon(v) := \int_0^v f_\varepsilon(v) dv.$$

- Let $W(v) := \frac{1}{2} v^2 \log|v| - \frac{1}{4} v^2 = \int_0^v f(v) dv$. Then,

$$W_\varepsilon(v) = \frac{1}{2}\left[\log(\varepsilon) + \mathcal{O}(1)\right] v^2 \leq 0, \quad |v| \leq \varepsilon$$

and

$$W_\varepsilon(v) = W(v) + C\varepsilon^2, \quad |v| \geq \varepsilon.$$

Step 2: uniform energy estimates

- Use energy conservation

$$\|v^\varepsilon(t)\|_{L^2} = \|v_0\|_{L^2}, \quad E_\varepsilon(v^\varepsilon(t)) = E_\varepsilon(v_0), \quad \text{for} \quad \text{every } t \in [-T_\varepsilon, T_\varepsilon],$$

  where

$$E_\varepsilon(v) := \frac{1}{2}\int_{\mathbb{R}} (v_x)^2 dx - \int_{\mathbb{R}} W_\varepsilon(v) dx, \quad W_\varepsilon(v) := \int_0^v f_\varepsilon(v) dv.$$

- Let $W(v) := \frac{1}{2}v^2 \log|v| - \frac{1}{4}v^2 = \int_0^v f(v)dv$. Then,

$$W_\varepsilon(v) = \frac{1}{2}\left[\log(\varepsilon) + \mathcal{O}(1)\right] v^2 \leq 0, \quad |v| \leq \varepsilon$$

  and

$$W_\varepsilon(v) = W(v) + C\varepsilon^2, \quad |v| \geq \varepsilon.$$

- There is $C > 0$ such that $[W_\varepsilon(v)]_+ \leq C|v|^3$ and the approximating solutions are extended to the global solutions $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$ such that

$$\|v^\varepsilon(t)\|_{H^1} + \|(v^\varepsilon(t))^2 \log|v^\varepsilon(t)|\|_{L^1} \leq C(v_0).$$

Step 3: passage to the limit

Assume that $v_0 \in X \subset H^1(\mathbb{R})$. Then $E_\varepsilon(v_0) < \infty$ and $E(v_0) < \infty$.

- Since $|W_\varepsilon(v)| \leq |W(v)| + Cv^2$ for every $v \in \mathbb{R}$, by Lebesque's dominated convergence theorem, we have

$$E_\varepsilon(v_0) \to E(v_0) \quad \text{as } \varepsilon \to 0, \quad \text{for every } v \in X.$$

Step 3: passage to the limit

Assume that $v_0 \in X \subset H^1(\mathbb{R})$. Then $E_\varepsilon(v_0) < \infty$ and $E(v_0) < \infty$.

- Since $|W_\varepsilon(v)| \leq |W(v)| + Cv^2$ for every $v \in \mathbb{R}$, by Lebesque's dominated convergence theorem, we have

$$E_\varepsilon(v_0) \to E(v_0) \quad \text{as } \varepsilon \to 0, \quad \text{for every } v \in X.$$

- The sequence v^ε is bounded in space $L^\infty(\mathbb{R}, X)$, whereas the sequence v_t^ε is bounded in space $L^\infty(\mathbb{R}, H^{-2}(\mathbb{R}))$.

- From Arzela–Ascoli Theorem, there exist $v \in L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ and a subsequence v^ε such that

$$v^\varepsilon \to v \quad \text{strongly in } L^\infty_{\mathrm{loc}}(\mathbb{R}, H^s_{\mathrm{loc}}(\mathbb{R})) \quad \text{as } \varepsilon \to 0, \quad \text{for all } s < 1$$

and for almost every $x \in \mathbb{R}$ and every $t \in \mathbb{R}$,

$$v^\varepsilon(x,t) \to v(x,t) \quad \text{as } \varepsilon \to 0.$$

Step 3: passage to the limit

- By Fatou's lemma, $v \in L^{\infty}(\mathbb{R}, X)$ with

$$\|v(\tau)\|_{L^2} \leq \lim_{\varepsilon \to 0} \|v^{\varepsilon}(t)\|_{L^2} = \|v_0\|_{L^2}$$

and

$$E(v(\tau)) \leq E(v_0), \quad \text{for all } \tau \in \mathbb{R}.$$

Step 3: passage to the limit

- By Fatou's lemma, $v \in L^\infty(\mathbb{R}, X)$ with

$$\|v(\tau)\|_{L^2} \leq \lim_{\varepsilon \to 0} \|v^\varepsilon(t)\|_{L^2} = \|v_0\|_{L^2}$$

and

$$E(v(\tau)) \leq E(v_0), \quad \text{for all } \tau \in \mathbb{R}.$$

- The limiting function $v \in L^\infty(\mathbb{R}, X)$ is a weak global solution of the log–KdV equation

$$\partial_t v + \partial_x (v \log|v|) + \partial_x^3 v = 0$$

in the sense

$$\int_{\mathbb{R}} \left[\langle v, \psi \rangle_{L^2} \phi'(t) + \langle v, \psi''' \rangle_{L^2} \phi(t)\right] dt + \int_{\mathbb{R}} \int_{\mathbb{R}} f(v) \psi'(x) \phi(t) dx dt = 0,$$

where ψ and ϕ are any test functions. $\square$

Uniqueness and global well-posedness

Lemma: *Assume that a solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation satisfies the additional condition*

$$\partial_x \log|v| \in L^\infty([-t_0, t_0] \times \mathbb{R}).$$

Then, the solution v is unique for every $t \in (-t_0, t_0)$, depends continuously on the initial data $v_0 \in X$, and satisfies $\|v(t)\|_{L^2} = \|v_0\|_{L^2}$ and $E(v(t)) = E(v_0)$ for all $t \in (-t_0, t_0)$.

Uniqueness and global well-posedness

Lemma: *Assume that a solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation satisfies the additional condition*

$$\partial_x \log|v| \in L^\infty([-t_0, t_0] \times \mathbb{R}).$$

Then, the solution v is unique for every $t \in (-t_0, t_0)$, depends continuously on the initial data $v_0 \in X$, and satisfies $\|v(t)\|_{L^2} = \|v_0\|_{L^2}$ and $E(v(t)) = E(v_0)$ for all $t \in (-t_0, t_0)$.

- $\partial_x \log|v|$ is unbounded as $|x| \to \infty$ for the Gaussian solitary wave.
- Nonlinear orbital stability of Gaussian solitary wave is conditional that the global solution $v \in L^\infty(\mathbb{R}, X)$ is unique and depends continuously on the initial data $v_0 \in X$.

Proof of uniqueness

Suppose that v and u are two local solutions of the log–KdV equation starting with the same initial data v_0. Set $w := v - u$ such that $w|_{t=0} = 0$. Then w satisfies

$$w_t + w_{xxx} + (v \log|v| - u \log|u|)_x = 0,$$

from which we obtain

$$\frac{d}{dt}\frac{1}{2}\|w\|_{L^2}^2 = -\int_{\mathbb{R}} (v_x \log|v| - u_x \log|u|) w dx,$$

Proof of uniqueness

Suppose that v and u are two local solutions of the log–KdV equation starting with the same initial data v_0. Set $w := v - u$ such that $w|_{t=0} = 0$. Then w satisfies

$$w_t + w_{xxx} + (v \log|v| - u \log|u|)_x = 0,$$

from which we obtain

$$\frac{d}{dt}\frac{1}{2}\|w\|_{L^2}^2 = -\int_{\mathbb{R}} (v_x \log|v| - u_x \log|u|) w dx,$$

By using the bound for the logarithimic nonlinearity,

$$|\log|v| - \log|u|| \leq \frac{|v-u|}{\min(|v|,|u|)},$$

we obtain

$$\left|\frac{d}{dt}\|w\|_{L^2}^2\right| \leq 3\left(\left\|\frac{v_x}{v}\right\|_{L^\infty} + \left\|\frac{u_x}{u}\right\|_{L^\infty}\right)\|w\|_{L^2}^2.$$

Gronwall's inequality yields $\|w(t)\|_{L^2}^2 = 0$, $t \in (-t_0, t_0)$. □

Spectral stability

Let $v_0 = e^{\frac{2-x^2}{4}}$ be the Gaussian wave. If $v = v_0(x) + V(x)e^{\lambda t}$, we arrive to the linear eigenvalue problem

$$\partial_x L V = \lambda V, \quad L = -\partial_x^2 - \frac{3}{2} + \frac{x^2}{4}.$$

Since $\sigma(L) = \{n-1, \quad n \in \mathbb{N}_0\}$, spectral stability of the Gaussian wave v_0 follows from an adaptation of recent works:

- T. Kapitula, A. Stefanov, *Stud. Appl. Math.*, in press (2014).
- D.P., in *Spectral analysis, stability, and bifurcation in modern nonlinear physical systems* (Wiley–ISTE, 2014).

Theorem 2 (R. Carles–D.P., 2014)

The spectrum of $\partial_x L$ in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ such that $0 < \omega_1 < \omega_2 < \ldots$ and $\omega_n \to \infty$ as $n \to \infty$. The eigenfunctions for nonzero eigenvalues are smooth in x but decay algebraically as $|x| \to \infty$.

Further remarks

- Because the spectrum of $\partial_x L$ is purely discrete, no asymptotic stability result can hold for Gaussian solitary waves.
- This agrees with the result of Cazenave for the log–NLS equation: the L^p norms at the solution v for any $p \geq 2$ including $p = \infty$ may not vanish as $t \to \infty$ (or in a finite time).

Further remarks

- Because the spectrum of $\partial_x L$ is purely discrete, no asymptotic stability result can hold for Gaussian solitary waves.
- This agrees with the result of Cazenave for the log–NLS equation: the L^p norms at the solution v for any $p \geq 2$ including $p = \infty$ may not vanish as $t \to \infty$ (or in a finite time).
- Nonlinear analysis of perturbations to the Gaussian solitary wave is problematic. If $v(x,t) := v_0(x) + w(x,t)$ is set, then

$$w_t = \partial_x L w - \partial_x N(w),$$

where

$$N(w) := w \log\left(1 + \frac{w}{v_0}\right) + v_0 \left[\log\left(1 + \frac{w}{v_0}\right) - \frac{w}{v_0}\right].$$

However, w/v_0 may grow like an inverse Gaussian function of x.

Proof of spectral stability

The linear eigenvalue problem

$$AV = \lambda V, \quad A := \partial_x L = -\partial_x^3 + \frac{1}{4}(x^2 - 6)\partial_x + \frac{1}{2}x,$$

can be written in the equivalent form with the Fourier transform

$$\hat{A}\hat{V} = \lambda \hat{V}, \quad \hat{A} = \frac{i}{4}k\left(-\partial_k^2 + 4k^2 - 6\right).$$

with the natural choice $\lambda = \frac{i}{4}E$.

Eigenfunctions of A are defined in the domain $X_A := D(A) \cap \dot{H}^{-1}(\mathbb{R})$,

$$D(A) = \left\{u \in H^3(\mathbb{R}) : \quad x^2\partial_x u \in L^2(\mathbb{R}), \quad xu \in L^2(\mathbb{R})\right\}.$$

In the Fourier form, the domain X_A becomes

$$\hat{X}_A = \left\{\hat{u} \in H^1(\mathbb{R}) : \quad k\partial_k^2 \hat{u} \in L^2(\mathbb{R}), \quad k^3\hat{u} \in L^2(\mathbb{R}), \quad k^{-1}\hat{u} \in L^2(\mathbb{R})\right\}.$$

Proof of spectral stability

The linear eigenvalue problem is

$$\frac{d^2\hat{u}}{dk^2} + \left(\frac{E}{k} + 6 - 4k^2\right)\hat{u}(k) = 0, \quad k \in \mathbb{R}.$$

- As $k \to 0$, two linearly independent solutions exist

$$\hat{u}_1(k) = k + \mathcal{O}(k^2), \quad \hat{u}_2(k) = 1 + \mathcal{O}(k\log(k)).$$

The second solution does not belong to $\hat{X}_A$.

- As $|k| \to \infty$, the decaying solution satisfies

$$\hat{u}(k) = ke^{-k^2}\left(1 + \mathcal{O}(|k|^{-1})\right).$$

The shooting problem is over-determined.

Proof of spectral stability

- The way around is the weak piecewise definition of the eigenfunction:

$$\hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

Proof of spectral stability

- The way around is the weak piecewise definition of the eigenfunction:

$$\hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

  where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

- For $\hat{u}_+$, we set $\hat{u}_+(k) = k^{1/2}\hat{v}_+(k)$ and obtain

$$k^{1/2}\left(-\frac{d^2}{dk^2} + 4k^2 - 6\right)k^{1/2}\hat{v}_+(k) = E\hat{v}_+(k), \quad k \in (0,\infty),$$

  which is now in the symmetric form. Hence $E \in \mathbb{R}$.

Proof of spectral stability

- The way around is the weak piecewise definition of the eigenfunction:

$$\hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

  where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

- For $\hat{u}_+$, we set $\hat{u}_+(k) = k^{1/2}\hat{v}_+(k)$ and obtain

$$k^{1/2}\left(-\frac{d^2}{dk^2} + 4k^2 - 6\right)k^{1/2}\hat{v}_+(k) = E\hat{v}_+(k), \quad k \in (0,\infty),$$

  which is now in the symmetric form. Hence $E \in \mathbb{R}$.

- For $E = 0$, we have $\hat{v}_+ = k^{1/2}e^{-k^2} > 0$ for $k > 0$. By Sturm's Theorem, the set of eigenvalues $\{E_n\}_{n\in\mathbb{N}_0}$ satisfies $0 = E_0 < E_1 < E_2 < \ldots$ and $E_n \to \infty$ as $n \to \infty$. $\square$

Numerical illustration

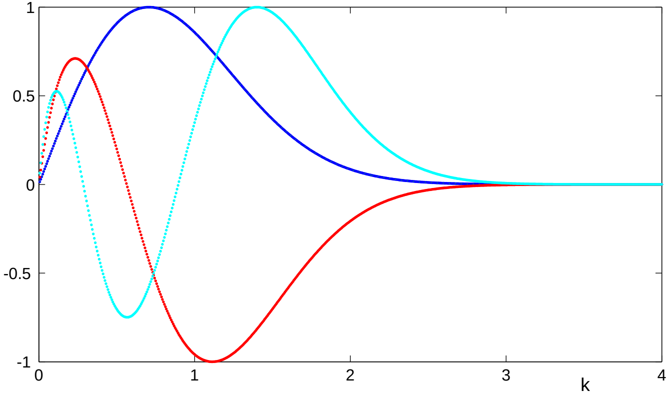

Figure : Eigenfunctions $\hat{u}$ of the spectral problem versus k for the first three eigenvalues $E_0 = 0$, $E_1 \approx 5.411$, and $E_2 \approx 12.308$.

Linear orbital stability

Gaussian wave $v_0 = e^{\frac{2-x^2}{4}}$ is a critical point of the energy $E(v)$: $E'(v_0) = 0$. The Hessian operator at the critical point $v_0 = e^{\frac{2-\xi^2}{4}}$ is $L = E''(v_0) = -\partial_x^2 - \frac{3}{2} + \frac{x^2}{4}$. The spectrum of L consists of simple eigenvalues at integers $n-1$, where $n \in \mathbb{N}_0$ (the set of natural numbers including zero).

Consider the time evolution of the perturbation u to v_0:

$$u_t = \partial_x L u, \quad u(0) = u_0.$$

Theorem 3 (G.James–D.P., 2014)

The solitary wave v_0 is linearly orbitally stable in space $H^1(\mathbb{R})$ in the following sense. For every $u_0 \in D(\partial_x L)$ such that $\langle v_0, u_0 \rangle_{L^2} = 0$, there exists constant $C(u_0)$ such that

$$\|u(t)\|_{H^1} \leq C(u_0), \quad t \in \mathbb{R}.$$

Symplectic decomposition

We know that $\partial_x L$ has a double zero eigenvalue because

$$L v_0' = 0, \quad \partial_x L v_0 = -v_0',$$

and no $u \in D(\partial_x L)$ exists in $\partial_x L u = v_0$ because $\|v_0\|_2^2 \neq 0$.

Using the decomposition

$$u(x,t) = a(t)\, v_0'(x) + b(t)\, v_0(x) + y(x,t)$$

with $\langle v_0, y \rangle_{L^2} = 0$ and $\langle \partial_x^{-1} v_0, y \rangle_{L^2} = 0$, we obtain

$$\frac{da}{dt} + b = 0, \quad \frac{db}{dt} = 0, \quad \frac{\partial y}{\partial t} = \partial_x L y.$$

If $\langle v_0, u_0 \rangle_{L^2} = 0$, then $b(t) = b(0) = 0$ and $a(t) = a(0)$.

Proof of linear orbital stability

Because v_0 and v_0' are eigenvectors of L for the negative and zero eigenvalues, L is strictly positive definite on ${v_0}^{\perp} \cap {v_0'}^{\perp} \subset L^2(\mathbb{R})$.

As a result, $\|y\|_L = \langle Ly, y \rangle_{L^2}^{1/2}$ defines a norm (equivalent to a weighted H^1-norm).

From the energy balance,

$$\frac{d}{dt}\frac{1}{2}\|y\|_L^2 = \langle Ly, \partial_t y \rangle_{L^2} = \langle Ly, \partial_x Ly \rangle_{L^2} = 0,$$

we obtain the Lyapunov stability of the zero equilibrium $y = 0$ in the constrained space $\langle v_0, y \rangle_{L^2} = 0$ and $\langle \partial_x^{-1} v_0, y \rangle_{L^2} = 0$. $\square$