Selection of the ground state on compact metric graphs

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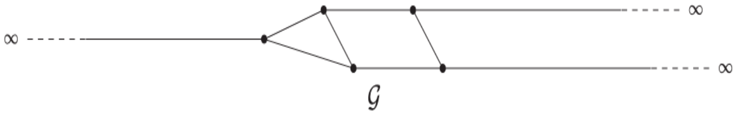
1. Metric graphs

In many problems (photonics, optics, river networks, bio networks), wave dynamics can be considered trapped on a graph Γ that consists of a set of edges $E = \{e_j\}$ connected at the set of vertices $V = \{v_j\}$.

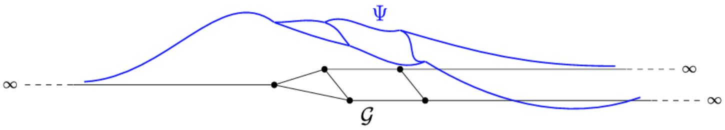


Compared to combinatorial graphs, edge lengths are important, whereas the angles between the edges at each vertex and the directions of orientation of each edge does not play any role. We can introduce a metric space on a metric graph, e.g. $L^p(\Gamma)$ defined componentwise on edges E with the norm

$$\|\Psi\|_{L^p(\Gamma)} = \left(\sum_{e_j \in E} \|\psi_j\|^p_{L^p(e_j)} \right)^{1/p},$$

for a componentwise defined function $\Psi(x) : \Gamma \to \mathbb{C}$.

Laplace operator on a metric graph



One can define Laplace operator $\Delta_\Gamma : \mathrm{dom}(\Delta_\Gamma) \subset L^2(\Gamma) \to L^2(\Gamma)$ componentwise on edges $E = \{e_j\}$:

$$(\Delta_\Gamma \Psi)(x) = \frac{d^2\psi_j}{dx^2}, \quad x \in e_j.$$

However, we need boundary conditions on vertices $V = \{v_j\}$ which connect components of Ψ on every edge incident to a particular vertex v. The *natural* conditions are called Neumann–Kirchhoff boundary conditions:

$$\begin{cases} \psi_e(v) = \psi_{e'}(v) & \text{for every } e, e' \prec v, \\ \sum_{e \prec v} \partial\psi_e(v) = 0, & \end{cases}$$

where $\partial\psi_e(v)$ denotes outward derivative of ψ_e at v (which is $\pm\frac{d\psi_e}{dx}(v)$).

PDEs on a metric graph

The Laplace operator is self-adjoint on $L^2(\Gamma)$ subject to the Neumann–Kirchhoff (NK) boundary conditions:

$$\langle -\Delta_\Gamma \Psi, \Phi \rangle_{L^2(\Gamma)} = \langle \nabla_\Gamma \Psi, \nabla_\Gamma \Phi \rangle_{L^2(\Gamma)} = \langle \Psi, -\Delta_\Gamma \Phi \rangle_{L^2(\Gamma)}$$

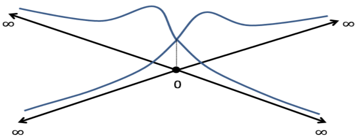
for every $\Psi, \Phi \in \mathrm{dom}(\Delta_\Gamma) = \{H^2(\Gamma) : \text{ NK conditions}\}$. Similarly, one can define the form domain of Δ_Γ by $H^1_C(\Gamma) = \{H^1(\Gamma) : \text{continuity conditions}\}$.

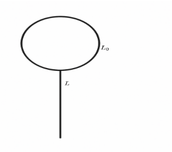
Many nonlinear PDEs can be expressed by using the Laplace operator Δ_Γ:

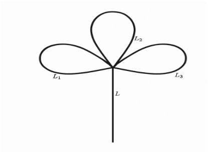
$$\begin{aligned}
&\text{(NLS)} && i\partial_t \Psi = -\Delta_\Gamma \Psi + |\Psi|^p \Psi, \\
&\text{(Fisher–KPP)} && \partial_t \Psi = \Delta_\Gamma \Psi + \Psi(1-\Psi), \\
&\text{(sine-Gordon)} && \partial_t^2 \Psi = \Delta_\Gamma \Psi + \sin(\Psi), \\
&\text{(Boussinesq)} && i\partial_t \Psi = (1-\Delta_\Gamma)^{-1} \left(\Delta_\Gamma \Psi + \Psi^2 \right).
\end{aligned}$$

Some nonlinear PDEs are expressed by the first-order operators (Dirac) and third-order operators (KdV). See review:
A. Kairzhan, D. Noja, and D.E. Pelinovsky, Standing waves on quantum graphs, J. Phys. A: Math. Theor. 55 (2022) 243001 (51pp)

Examples: star graphs, tadpole and flower graphs, etc.







- Variational methods were developed by R. Adami, S. Dovetta, E. Serre, P. Tilli, A. Pankov, . . .
- Dynamical stability (PDE) methods were developed by J. A. Pava, N. Goloshchapova, M. Cavalcante, R. Plaza, A. Ardila, . . .
- Dispersive estimates were developed by V. Banica, L. Ignat, A. Grecu, F. Mehmeti, K. Ammari, S. Nicaise, . . .
- Period function and ODE methods were developed by A. Kairzhan, R. Marangell, D. Noja, S. Le Coz, H. Tavares, J. A. Pava,. . .
- Homogenization, bifurcation, asymptotic methods were developed by G. Berkolaiko, J. Marzuola, R. Goodman, G. Schneder, . . .

2. The Fisher–KPP model on metric graphs

The Fisher–KPP equation, the Keller–Segel systems, and other reaction–diffusion PDEs on a metric graph has been considered for modeling of populations, river networks, chemotaxis:

$$(\text{Fisher-KPP}) : \quad \partial_t u = \Delta_\Gamma u + u(1-u).$$

and

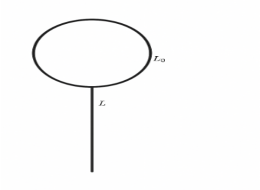
$$(\text{Keller–Segel}) : \quad \begin{cases} \partial_t u = \Delta_\Gamma u - \nabla_\Gamma \chi(u,v) \nabla_\Gamma v + f(u,v), \\ \tau \partial_t v = \Delta_\Gamma v + g(u,v). \end{cases}$$

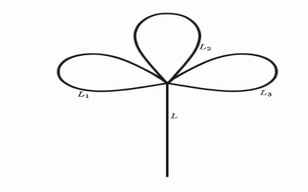
The recent mathematical studies include the following:

- Existence and stability of steady states on compact metric graphs - E. Yanagida (2001), Y. Ishii & K. Kurata (2021), Y. Ishii (2022)
- Bifurcations and asymptotic stability of constant states - H. Shemtaga, W. Shen, & S. Sukhtaiev (2024, 2025)
- Spreading speeds of a propagation front (slowing down) - W-T. L. Fan, W. Hu, & G. Terlov (2021)
- Trapping of the spreading front at vertices - S. Jimbo & Y. Morita (2019, 2021, 2024)

Main problem related to the Fisher–KPP equation

Let Γ be a compact metric graph with finitely many edges of finite lengths. We assume that Γ is connected and there is at least one pendant (an edge with a boundary vertex not connected to any other edge).





We use the Neumann–Kirchhoff conditions on interior vertices and the Dirichlet condition on boundary vertices.

$$\partial_t u = \Delta_\Gamma u + u(1-u) \quad \text{on } \Gamma.$$

$u = 0$ is a constant (trivial) state but $u = 1$ is no longer a valid state.

Question: existence, uniqueness, and asymptotic stability of the nonconstant (positive) states in dependence of the graph geometry (e.g. edge lengths).

The gradient system with the free energy

The Fisher–KPP equation is a gradient system with the energy

$$H(u) = \frac{1}{2}\int_\Gamma [(\nabla_\Gamma u)^2 - u^2]dx + \frac{1}{3}\int_\Gamma u^3 dx,$$

defined on

$$\mathcal{H}_0 = \left\{u \in H_0^1(\Gamma) : \quad u \geq 0\right\}.$$

If $u_0 \in \mathcal{H}_0$, then there exists a unique global solution $u \in C^0([0,\infty),\mathcal{H}_0)$ with initial data $u|_{t=0} = u_0$, which converges to a global attractor $u_* \in \mathcal{H}_0$ as

$$u_* = \lim_{t\to+\infty} u(t,\cdot).$$

The attractor $u_* \in \mathcal{H}_0$ is a critical point of $H(u)$, that is, a solution of the steady state problem

$$-\Delta_\Gamma u = u(1-u), \quad u \in \mathrm{dom}(\Delta_\Gamma).$$

Main question is now to characterize the existence, uniqueness, and asymptotic stability of $u_* \neq 0$ in dependence of the graph geometry.

3. Main results

Let $\{L_j\}$ be the set of lengths of edges of the graph Γ.

1. There is a unique global attractor for every $u_0 \in \mathcal{H}_0$. The attractor is either trivial or nontrivial depending whether $\lambda_0(\Gamma) > 1$ or $\lambda_0(\Gamma) < 1$, where $\lambda_0(\Gamma)$ is the smallest eigenvalue of $-\Delta_\Gamma$ in $L^2(\Gamma)$.
2. There exists $L_0 > 0$ such that if $\max_j L_j < L_0$, then the trivial (zero) state is a global attractor and a minimizer of the energy $H(u)$ in $\mathcal{H}_0$.
3. For every j-th edge of length L_j, there is $L_j^{(0)} \in [0, \infty)$ such that for every $L_j > L_j^{(0)}$, a strictly positive state is a global attractor and a minimizer of the energy $H(u)$ in $\mathcal{H}_0$ at a negative level.
4. There exist $L_* > L_0$ such that if $L_{\min} := \min_j L_j > L_*$, then the unique, strictly positive state u_* satisfies

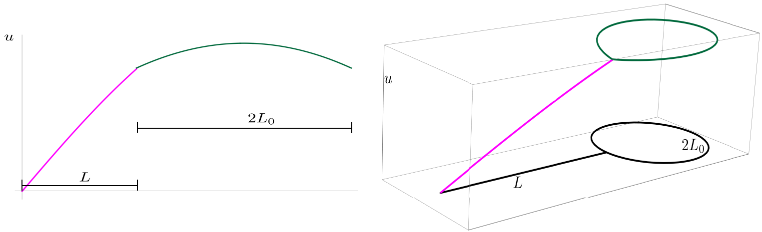
$$\|u_* - 1\|_{L^\infty(\Gamma_0)} \leq Ce^{-L_{\min}},$$

for some $C > 0$, where Γ_0 is the part of Γ without the pendants.

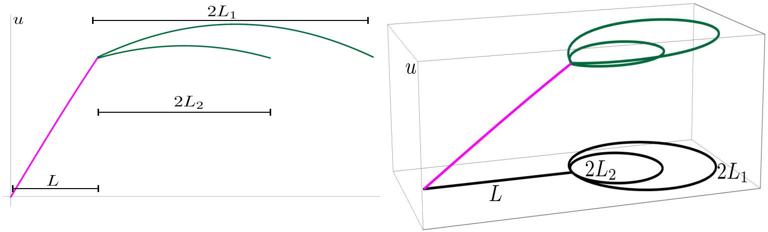
[R. Marangell, D.P., Indiana University Math. J. (2026), in print]

Illustration: positive states on the flower graphs

For the tadpole graph with a single loop, the unique positive state is



For the flower graph with two loops of different lengths, it is



4. Classical methods for the Fisher–KPP equation

The spectrum of $-\Delta_\Gamma$ in $L^2(\Gamma)$ consists of isolated positive eigenvalues, since Γ is compact. Furthermore, the Laplace equation associated with Δ_Γ enjoyes the same maximum principle on Γ as in an open region of $\mathbb{R}^d$.

As a result, we can use the classical methods of analysis:

- Linearization and the lowest eigenvalue of $-\Delta_\Gamma$.
- Variational methods for the energy $H(u)$.
- Comparison principle with subsolutions and supersolutions.

This gives the main results (1), (2), and (3). For the exponential localization (4), we develop a novel method based on the period function. The period function method also gives the precise construction of the ground state.

Classical methods: the lowest eigenvalue of $-\Delta_\Gamma$

Lemma (Berkolaiko–Kuchment; Latushkin–Sukhtaiev; ...)

Let the set of lengths $\{L_j\}$ *be fixed except for one edge parameterized as* $[0,L]$. *Let* $\lambda(\Gamma) \geq 0$ *be a simple eigenvalue of* $-\Delta_\Gamma$ *in* $L^2(\Gamma)$ *with the corresponding eigenfunction* $\Psi : \Gamma \to \mathbb{R}$ *in* $\mathrm{dom}(\Delta_\Gamma)$. *Then,* $\lambda(\Gamma)$ *is a* C^1 *function of* L *satisfying*

$$\frac{d\lambda(\Gamma)}{dL} = -|\psi'(L)|^2 - \lambda(\Gamma)|\psi(L)|^2 \leq 0,$$

where $\psi(x) : [0,L] \to \mathbb{R}$ *is the component of* Ψ *on the edge* $[0,L]$.

Corollary

Let $\lambda_0(\Gamma)$ *be the lowest eigenvalue of* $-\Delta_\Gamma$. *Then,* $\Psi(x) > 0$ *for all* $x \in \Gamma$ *except for the boundary vertices (under Dirichlet condition) and* $\frac{d\lambda_0(\Gamma)}{dL} < 0$.

Corollary

The trivial state 0 *in* $\partial_t u = \Delta_\Gamma u + u(1-u)$ *is stable if* $\lambda_0(\Gamma) \geq 1$ *and unstable if* $\lambda_0(\Gamma) < 1$.

Classical methods: the energy minimization

Lemma

If $\lambda_0(\Gamma) \geq 1$, then the infimum of $H(u)$ in $\mathcal{H}_0$ is attained at $u = 0$, $H(0) = 0$.
If $\lambda_0(\Gamma) < 1$, then the infimum of $H(u)$ in $\mathcal{H}_0$ is attained at u_*, $H(u_*) < 0$, where $u_* > 0$ for all points of Γ except for boundary vertices.

This follows from the Rayleigh quotient which yields

$$\int_\Gamma (\nabla_\Gamma u)^2 dx \geq \lambda_0(\Gamma) \|u\|^2_{L^2(\Gamma)}, \quad \forall u \in H^1_0(\Gamma).$$

- If $\lambda_0(\Gamma) \geq 1$, then

$$H(u) \geq \frac{1}{2} \int_\Gamma [(\nabla_\Gamma u)^2 - u^2] dx \geq (\lambda_0(\Gamma) - 1) \|u\|^2_{L^2(\Gamma)},$$

- If $\lambda_0(\Gamma) < 1$, then $H(u)$ is bounded from below:

$$H(u) \geq -\frac{1}{2} \int_\Gamma u^2 dx + \frac{1}{3} \int_\Gamma u^3 dx \geq -\frac{1}{6} \sum_j L_j.$$

Dependence of the ground state on the edge lengths

By subsolutions and supersolutions, the unique attractor in $\mathcal{H}_0$ is the trivial solution $u = 0$ if $\lambda_0(\Gamma) \geq 1$ and a nontrivial solution $u_* > 0$ if $\lambda_0(\Gamma) < 1$.

Corollary

There exists $L_0 > 0$ such that if $\max_j L_j < L_0$, then $u = 0$ is globally asymptotically stable for initial data $u_0 \in \mathcal{H}_0$.

This follows the uniform scaling of eigenvalues $\lambda(\Gamma)$. If $L_j = L\ell_j$ with L-independent $\{\ell_j\}$, then $\lambda(\Gamma) = L^{-2}\lambda(\tilde{\Gamma})$ with L-independent $\lambda(\tilde{\Gamma})$.

Corollary

For every j-th edge of length L_j, $\exists L_j^{(0)} \in [0,\infty)$: for every $L_j > L_j^{(0)}$ with this j, $u = u_* \geq 0$ is globally asymptotically stable for initial data $u \in \mathcal{H}_0 \backslash \{0\}$.

This follows from the monotone decrease of $\lambda_0(\Gamma)$ and the convergence $\lambda_0(\Gamma) \to 0 = \inf \sigma_{\text{ess}}(-\Delta_\Gamma)$ in $L^2(0,\infty)$ as $L_j \to \infty$ for at least one L_j.

5. Novel methods based on the period function

Since Γ consists of edges $\{e_j\}$ (which are 1D intervals), we can consider the integral curves of the second-order equation

$$u''(x) + u(x) - u(x)^2 = 0, \quad u(x) : \mathbb{R} \to \mathbb{R}$$

to construct piecewisely smooth solutions to

$$-\Delta_\Gamma u - (1-u)u = 0, \quad \psi \in \mathrm{dom}(\Delta_\Gamma).$$

We will work with $\tilde{u} = 1 - u$ satisfying

$$\tilde{u}''(x) - \tilde{u}(x) + \tilde{u}(x)^2 = 0, \quad \tilde{u}(x) : \mathbb{R} \to \mathbb{R},$$

so that $\tilde{u} = 1$ at the boundary vertices and $\tilde{u} \in (0,1)$ everywhere in Γ.

Integral curves on the phase plane $(\tilde{u}, \tilde{v})$ correspond to constant values of the first-order invariant

$$E(\tilde{u}, \tilde{v}) = \tilde{v}^2 - \tilde{u}^2 + \frac{2}{3}\tilde{u}^3, \quad \tilde{v} := \frac{d\tilde{u}}{dx}. \tag{1}$$

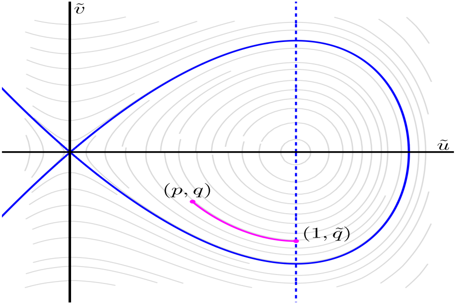
Period function I

Let $(p, q) \in (0, 1) \times (-\infty, 0)$ be a point on the phase plane $(\tilde{u}, \tilde{v})$ and consider an integral curve connecting $(1, \tilde{q})$ and (p, q), where

$$E(p,q) = q^2 - p^2 + \frac{2}{3}p^3 = \tilde{q}^2 - \frac{1}{3}$$

The period function $T(p, q)$ is defined by

$$T(p,q) = \int_p^1 \frac{du}{v}, \quad v := \sqrt{E(p,q) + u^2 - \frac{2}{3}u^3}.$$



Main results about $T(p,q)$

Lemma

For every $(p,q) \in (0,1) \times (-\infty, 0)$, *we have*

$$\frac{\partial T}{\partial p} < 0, \qquad \frac{\partial T}{\partial q} > 0.$$

Two asymptotic limits near the saddle and center points are also important:

- As $(p,q) \to (0,0)$, we have

$$T(p,q) = -\ln\left(\frac{p-q}{12}\right) - x_0 + \mathcal{O}(p),$$

where $x_0 = 2\text{arccosh}\left(\frac{\sqrt{3}}{\sqrt{2}}\right)$.

- As $(p,q) \to (1,0)$, we have

$$T(p,q) = \arcsin\frac{1-p}{\sqrt{(1-p)^2+q^2}} + \mathcal{O}(|1-p|).$$

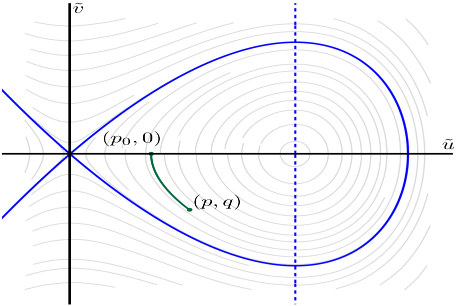
Period function II

Let $(p, q) \in (0, 1) \times (-\infty, 0)$ be a point on the phase plane $(\tilde{u}, \tilde{v})$ and consider an integral curve connecting (p, q) and $(p_0, 0)$, where

$$E(p, q) = q^2 - p^2 + \frac{2}{3}p^3 = -p_0^2 + \frac{2}{3}p_0^3.$$

The period function $T_0(p, q)$ is defined by

$$T_0(p, q) = \int_{p_0}^{p} \frac{du}{v}, \quad v := \sqrt{E(p, q) + u^2 - \frac{2}{3}u^3}.$$



Main results about $T_0(p,q)$

Lemma

For every $(p,q) \in (0,1) \times (-\infty, 0)$, *we have*

$$\frac{\partial T_0}{\partial q} < 0.$$

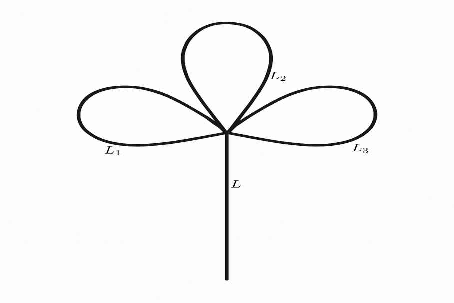
However, $T_0(p,q)$ is more difficult for analysis:

- $\frac{\partial T_0}{\partial p} < 0$ only for $p \in (0, \frac{1}{2})$ and is not monotone if $p \in (\frac{1}{2}, 1)$.
- As $(p,q) \to (1,0)$, we only have the limit

  $$\lim_{(p,q)\to(1,0)} T_0(p,q) = \frac{\pi}{2} - \arcsin \frac{1-p}{\sqrt{(1-p)^2+q^2}},$$

  but not the remainder term.

Example: ground state on the symmetric flower graph



Let $(p,q) \in (0,1) \times (-\infty, 0)$ be parameters for the boundary conditions $p = \tilde{u}(L)$ and $q = \tilde{u}'(L)$ at the vertex. The steady state is given by the identical and even functions $\tilde{u}_j = \tilde{u}_0$ satisfying the boundary conditions $\tilde{u}_0(\pm L_0) = p$ and $\tilde{u}_0'(\pm L_0) = \mp \frac{q}{2N}$. The existence of the steady state is equivalent to finding a root (p,q) of the system of two equations:

$$T(p,q) = L, \qquad T_0\left(p, \frac{q}{2N}\right) = L_0.$$

Uniqueness of the ground state

Here is the equivalent system of equations:

$$T(p,q) = L, \qquad T_0\left(p, \frac{q}{2N}\right) = L_0,$$

for a given point $(L, L_0) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Theorem

There exists a simply connected region $\Omega \in \mathbb{R}^+ \times \mathbb{R}^+$ such that the positive steady state exists for every $(L, L_0) \in \Omega$ and is unique.

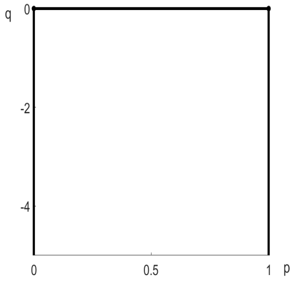
The result follows from the positivity of the Jacobian of the transformation $(p,q) \mapsto (L, L_0)$:

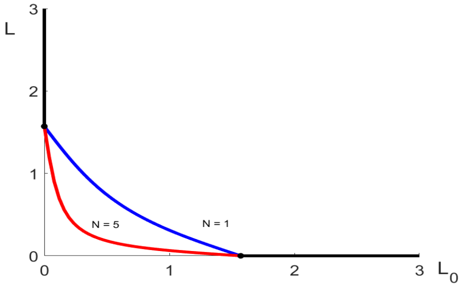
$$\frac{\partial T(p,q)}{\partial p} \frac{\partial T_0(p, \frac{q}{2N})}{\partial q} - \frac{\partial T(p,q)}{\partial q} \frac{\partial T_0(p, \frac{q}{2N})}{\partial p} > 0.$$

Existence of the ground state

Let $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^+$ be the existence region in (L, L_0). Then

- $L_0 = 0 : \Rightarrow [0, L]$ is an interval with Dirichlet and Neumann conditions. Then, $q = 0$ and $L > \frac{\pi}{2}$ for existence.
- $L = 0 : \Rightarrow [0, L_0]$ is an interval with Dirichlet and Neumann conditions. Then, $p = 1$ and $L_0 > \frac{\pi}{2}$ for existence.
- The lower boundary appears in the singular limit $(p, q) \to (1, 0)$, where $\cot(L) = 2N \tan(L_0)$. It is the same as $\lambda_0(\Gamma) = 1$ from classical theory.





6. The limit of long graphs: graphs inside out

Theorem

There exist $L_* > L_0$ such that if $L_{\min} := \min_j L_j > L_*$, then the positive ground state u_* is unique and satisfies

$$\|u_* - 1\|_{L^\infty(\Gamma_0)} \leq Ce^{-L_{\min}},$$

where C is a positive constant and $\Gamma_0 = \Gamma \backslash P_0$ without the set of pendants P_0.

Let V_0 be the subset of interior vertices $V = \{v_j\}$ which are boundaries between the set of pendants P_0 and Γ_0. Denote $p_j = \tilde{u}(v_j)$ at $v_j \in V_0$:

$$\begin{cases} -\Delta_{\Gamma_0}\tilde{u} + \tilde{u} = \tilde{u}^2 \ \text{ in } \Gamma_0, \\ \tilde{u} \ \text{ satisfies NK conditions on } \ V\backslash V_0, \\ \tilde{u}(v_j) = p_j \geq 0, \quad v_j \in V_0, \end{cases} \tag{in}$$

and

$$\begin{cases} -\tilde{u}''(x) + \tilde{u}(x) = \tilde{u}(x)^2 \ \text{ in } P_0, \\ \tilde{u}(\partial P_0) = 1, \\ \tilde{u}(v_j) = p_j \geq 0, \quad v_j \in V_0, \end{cases} \tag{out}$$

where we recall again that $\tilde{u} = 1 - u$.

Dirichlet-to-Neumann map I

For

$$\left\{ \begin{array}{l} -\Delta_{\Gamma_0}\tilde{u} + \tilde{u} = \tilde{u}^2 \;\; \text{in } \Gamma_0, \\ \tilde{u} \;\; \text{satisfies NK conditions on} \;\; V \backslash V_0, \\ \tilde{u}(v_j) = p_j \geq 0, \quad v_j \in V_0, \end{array} \right. \qquad (in)$$

there exist $C_0 > 0$, $p_0 > 0$, $L_* > 0$ such that if $L_{\min} = \min_j L_j > L_*$, then for every $\vec{p}$ such that $\|\vec{p}\| \leq p_0$, there is a unique solution $\tilde{u} \in \mathrm{dom}(\Delta_{\Gamma_0})$ satisfying

$$\|\tilde{u}\|_{L^\infty(\Gamma_0)} \leq C_0 \|\vec{p}\|$$

and

$$|q_j - d_j p_j| \leq C_0 \left(\|\vec{p}\| e^{-L_{\min}} + \|\vec{p}\|^2 \right), \quad v_j \in V_0,$$

where q_j is the Neumann data (the sum of outward derivatives from Γ_0) at the vertex v_j and d_j is the degree of the vertex $v_j \in V_0$.

G. Berkolaiko, J.L. Marzuola, and D.E. Pelinovsky, Edge-localized states on quantum graphs in the limit of large mass, Annales de l'Institut Henri Poincare C, Analyse Non Lineaire 38, 1295-1335 (2021)

Dirichlet-to-Neumann map II

For

$$\begin{cases} -\tilde{u}''(x) + \tilde{u}(x) = \tilde{u}(x)^2 \quad \text{in } P_0, \\ \tilde{u}(\partial P_0) = 1, \\ \tilde{u}(v_j) = p_j \geq 0, \quad v_j \in V_0. \end{cases} \qquad (out)$$

we denote $q_j = \tilde{u}'(L_j)$ and obtain from $T(p_j, q_j) = L_j$ in the limit of $L_{\min} = \min_j L_j \gg 1$ that

$$q_j = p_j - 12e^{-L_j - x_0} + \mathcal{O}(e^{-2L_j}), \quad x_0 = 2\text{arccosh}\left(\frac{\sqrt{3}}{\sqrt{2}}\right).$$

Bringing both Dirichlet–to–Neumann maps together in the NK conditions:

$$d_j p_j + \mathcal{O}(\|\vec{p}\| e^{-L_{\min}} + \|\vec{p}\|^2) + m_j p_j - 12 \sum_{\ell_j \to v_j} e^{-L_j - x_0} + \mathcal{O}(e^{-2L_{\min}}) = 0.$$

By the implicit function theorem, there exists a unique solution for small $\vec{p}$:

$$p_j = \frac{12}{d_j + m_j} \sum_{\ell_j \to v_j} e^{-L_j - x_0} + \mathcal{O}(e^{-2L_{\min}}).$$

7. Summary

We have considered the selection of the ground state on the compact metric graph with classical methods (linearization and the lowest eigenvalue of $-\Delta_\Gamma$, variational methods for the energy $H(u)$, and the comparison principle with subsolutions and supersolutions) and novel methods based on the period function for differential equations.

The period function methods give more precise infomation about construction of the ground state but their application is limited to either simplest graphs or to the asymptotic limit of long graphs. On the other hand, they work when the classical methods are not applicable.